Symmetries of Optical Phase Conjugation

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Various algebraic structures of degenerate four-wave mixing equations of optical phase conjugation are analyzed. Two approaches (the spinorial and the Lax-pair based), complementary to each other, are utilized for a systematic derivation of conserved quantities. Symmetry groups of both the equations and the conserved quantities are determined, and the corresponding generators are written down explicitly. Relation between these two symmetry groups is found. Conserved quantities enable the introduction of new methods for integration of the equations in the cases when the coupling Γ is either purely real or purely imaginary. These methods allow for both geometries of the process, namely the transmission and the reflection, to be treated on an equal basis. One approach to introduction of Hamiltonian and Lagrangian structures for the 4WM systems is explored, and the obstacles in successful implementation of that programe are identified. In case of real coupling these obstacles are removable, and full Hamiltonian and Lagrangian formulations of the initial system are possible.

I. INTRODUCTION

There is a short story prefacing the paper. The work on symmetries in optical phase conjugation started in early nineties by Predrag Stojkov and Milivoj Belić. It was interrupted by Predrag's leaving for America in 1992. During the stay of M. Belić at the Texas A&M University in 1995 and 1996, the problem and an early draft of the paper were brought to Marko's attention. At the time he was phasing out of quasicrystals, and was open to new ideas. Marko liked the problem and agreed to participate. He read the manuscript, made numerous changes, and suggested a new direction to it. Owing to his commitments for the sabbatical at Cornell University in 1995 and the visit to Israel in 1996, it was decided to postpone the serious work after he is back. However, during the Israeli visit Marko was diagnosed with the brain tumor. The paper is left essentially unchanged. It is dedicated to his memory.

Steady-state four-wave mixing (4WM) equations describing optical phase conjugation (OPC) in photorefractive (PR) crystals have been solved up to now in a number of ways [1–4]. A common feature of all solution methods is that, first, conserved quantities are determined, and then the number of equations is reduced. However, the determination of conserved quantities and the reduction of equations is usually performed in an ad hoc manner. Furthermore, the solution of the OPC equations in the two basic geometries of the process, the transmission geometry and the reflection geometry, is usually obtained using unrelated methods.

Apparent symmetries of wave equations have not been used up to now [5,6] to facilitate the analysis and the solution of the problem. In this work the symmetries of the equations and the integrals of motion are investigated and used to present a unified method for systematic derivation of conserved quantities, an equal treatment of both geometries, and the reduction in the number of independent variables [7]. Such an analysis allows for not only an easier handling of otherwise cumbersome and unrelated relations, but also for a deeper understanding of the physics of the process. Also, rudiments of a formal presentation of the problem along the lines of the theory of dynamical systems are presented.

The geometry of the process is simple. Three laser beams intersect within a piece of the PR crystal: two counterpropagating laser pumps A_1 and A_2, and a signal A_4. Owing to the PR effect, a fourth wave A_3 is generated inside the crystal, that counterpropagates to, and is the phase conjugate replica of the signal. There are two main channels along which the generation may proceed. In the first one, the signal wave builds a diffraction grating with the pump A_1. The other pump is diffracted off that grating and transmitted across the crystal into the PC wave A_3. This is the so-called transmission geometry (TG) of the process.

In the second channel the signal interferes with the pump A_2, and the beam A_1 is reflected off the grating into the PC wave. This is the reflection geometry (RG) of the 4WM process. It is assumed that all waves oscillate at the same frequency (the degenerate 4WM). Also, a steady-state is assumed, and all beams are approximated by plane waves.

The equations of interest are the slowly varying envelope wave equations describing 4WM in PR media [1]. In TG, they are of the form

$$IA_1' = \Gamma Q_T A_4,$$

$$
\begin{aligned}
IA_2' &= \bar{\Gamma}\bar{Q}_T A_3, \\
IA_3' &= -\bar{\Gamma}\bar{Q}_T A_2, \\
IA_4' &= -\Gamma Q_T A_1, \qquad (1.1)
\end{aligned}
$$

where $I = \sum_{i=1}^{4} |A_i|^2$ is the total intensity, Γ is the coupling constant (complex in general, but often real in PR media), $Q_T = A_1\bar{A}_4 + \bar{A}_2 A_3$ represents the diffraction grating amplitude for TG, the prime denotes spatial derivative along the propagation z direction, and the bar denotes complex conjugation.

In RG, the equations are given by

$$
\begin{aligned}
IA_1' &= -\Gamma Q_R A_3, \\
IA_2' &= -\bar{\Gamma}\bar{Q}_R A_4, \\
IA_3' &= -\bar{\Gamma}\bar{Q}_R A_1, \\
IA_4' &= -\Gamma Q_R A_2, \qquad (1.2)
\end{aligned}
$$

where the RG grating amplitude is given by $Q_R = A_1\bar{A}_3 + \bar{A}_2 A_4$.

In this paper both geometries are treated on an equal footing, using a unified RG-like notation:

$$
\begin{aligned}
B_1 &= A_1, \\
B_2 &= A_2, \\
B_3 &= A_4\Pi_\sigma + A_3\Pi_{-\sigma} = \begin{cases} A_4 \text{ in TG,} \\ A_3 \text{ in RG,} \end{cases} \\
B_4 &= A_3\Pi_\sigma + A_4\Pi_{-\sigma} = \begin{cases} A_3 \text{ in TG,} \\ A_4 \text{ in RG,} \end{cases} \qquad (1.3)
\end{aligned}
$$

where σ is the switching variable, that has value $+1$ for TG and -1 for RG, and $\Pi_{\pm\sigma} = (1 \pm \sigma)/2$ are the corresponding "projectors."

The "equations of motion" (EOM) are now

$$
\begin{aligned}
IB_1' &= \sigma\Gamma Q B_3, \\
IB_2' &= \sigma\bar{\Gamma}\bar{Q} B_4, \\
IB_3' &= -\bar{\Gamma}\bar{Q} B_1, \\
IB_4' &= -\Gamma Q B_2, \qquad (1.4)
\end{aligned}
$$

where the intensity is given by $I = \sum_{i=1}^{4} |B_i|^2$ and the grating amplitude by $Q = B_1\bar{B}_3 + \bar{B}_2 B_4$.

The analysis is organized as follows. Two methods to derive the integrals of motion (IOM) are discussed in Section II. The first method is based on the observation that 4WM EOM have a special symmetric form that is allowing an equivalent *spinorial* formulation. Such a form of EOM leads directly to the derivation of the full set of "regular" IOM as suitable bi-spinorial combinations. The symmetries of these IOM are the special unitary groups: $SU(2)$ for TG, and $SU(1,1)$ for RG. Initial spinor-like doublets of fields turn out to transform as the fundamental irreducible representations of these groups, thus justifying the name "spinors". It is indicated how they can be used to reduce the number of dynamical variables.

In the second method the Lax pair approach is utilized. The diadic products of the 4WM spinors are used as possible choices for the evolving member ($\mathcal{L}$) of the Lax pair problem. The traces of products of these matrices represent IOM. It was established that all higher order IOM are various combinations of the basic IOM already obtained by the spinorial approach. At the end of Section II two special cases ($\Gamma \in \mathbb{R}$ and $\Gamma \in i\mathbb{R}$) are considered in some detail.

In Section III attention is focused on the derivation of the symmetry groups of EOM, and the corresponding generators. The relation between these symmetries and the symmetries of IOM is discussed.

In Section IV the symmetries of IOM are used to write the solutions of EOM (for $\Gamma \in \mathbb{R}$) in terms of elementary transcendental functions. Then an alternative solution procedure is explored. In the last part of Section IV the $\Gamma \in i\mathbb{R}$ case is solved completely.

The possibility of introducing the Hamiltonian and Lagrangian description of 4WM EOM is explored in Section V. Section VI offers some conclusions and identifies open questions for future research.

II. INTEGRALS OF MOTION AND THEIR SYMMETRIES

A. Preliminaries

In this work the 4WM equations are treated as a dynamical system defined on the phase space $V :\equiv \tilde{V}/\rho \equiv \mathbb{R}^8$, with the time variable z. Here $\tilde{V}$ is the full sixteen-dimensional space $\tilde{V} :\equiv \mathbb{C}^8 \equiv \mathbb{R}^{16}$ with the complex coordinates $\{x^\mu\} \equiv \{B_i, \bar{B}_i\}$ and ρ is the equivalence relation (analyticity condition) satisfied by the 4WM system: $x^{i+4} = (x^i)^*$ [i.e. $\bar{B}_i = (B_i)^*$] for $i = \overline{1,4}$. The tangent space TV (the space of vector fields on V), is spanned by the *coordinate* basis $\{\partial_\mu\} \equiv \{\partial_i = \partial/\partial B_i; \bar{\partial}_i = \partial/\partial \bar{B}_i\}$. The dynamics on the space V is given by the trajectory $c : \mathbb{R}_z \to V$. It is described by the velocity vector-field $\vec{F} \in TV$:

$$
\vec{F} = \mu(\sigma B_3\partial_1 - B_2\partial_4) + \bar{\mu}(\sigma B_4\partial_2 - B_1\partial_3) + c.c. \qquad (2.1)
$$

where $\mu :\equiv \Gamma Q/I$. For a general function $f(z,x) \in C^1(\mathbb{R}_z \times V)$, the corresponding evolution equation is

$$
\left(\frac{df}{dz}\right) = (\partial_z + \vec{F})f. \qquad (2.2)
$$

In general, an integral of motion (IOM) $q(z,x)$ is a function that is constant along the trajectory c, i.e. $(\partial_z + \vec{F})q|_c = 0$ (*on-shell constancy*). Here the more restrictive definition of IOM is used: instead of an on-shell constancy, the condition of *off-shell* constancy [$(\partial_z + \vec{F})q = 0$ in whole V] is used. Also, only the integrals $q(x)$ without the explicit time-dependence are considered, leading to the defining equation

$$
\vec{F}q = 0. \qquad (2.3)
$$

There is no general procedure for finding IOM, and one has to take into account various specifics of the system at hand. For example, one may resort to the brute-force solution procedure, which is based on the observation that $\vec{F}$ is a linear differential operator of the zeroth degree of homogeneity (i.e. upon its action on some homogeneous function $P_s(x)$ of degree s, it produces another homogeneous function of the same degree s). This allows one to replace Eq. (2.3) by the set of infinitely many equations:

$$\vec{F}q_{(s)} = 0, \tag{2.4}$$

for $s \in \mathbb{N}$, where $q_{(s)}$ are the components of q with the fixed degree of homogeneity s ($q = \sum_{s=1}^{\infty} q_{(s)}$). In general, Eqs. (2.4) can be solved by a general ansatz (the summation over the repeated indices is assumed):

$$\begin{array}{rcl} q_{(1)} &=& \alpha_\mu x^\mu, \\ q_{(2)} &=& \alpha_{\mu\nu} x^\mu x^\nu, \\ & & \cdots \end{array} \tag{2.5}$$

which turns Eq. (2.4) into a set of conditions for the matrices α. After some algebra, one finds that there are no integrals of the first degree, and that there are several of the second and higher degrees.

B. Spinorial formalism

There exists a more elegant way to find integrals of motion of the second degree of homogeneity in the 4WM equations [6]. It is based on the fact that convenient pairs of columns ("spinors") can be formed:

$$|\psi_1\rangle = \begin{pmatrix} B_1 \\ B_3 \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} B_4 \\ -\sigma B_2 \end{pmatrix}. \tag{2.6}$$

Now the equations of motion (1.4) can be written as a pair of matrix equations:

$$|\psi_j\rangle' = \mathbf{m}\,|\psi_j\rangle\,, \quad (j = 1, 2) \tag{2.7}$$

where the "evolution matrix" $\mathbf{m}$ is

$$\mathbf{m} = \begin{pmatrix} 0 & \sigma\mu \\ -\bar{\mu} & 0 \end{pmatrix}, \tag{2.8}$$

and $\mu = \Gamma Q/I$. The matrix $\mathbf{m}$ is traceless and Hermitian (for TG) or skew-Hermitian (for RG), so it belongs to the $su(2)$ algebra for TG, or to the $su(1,1)$ for RG.

IOM are found using a simple Lemma:

• **<u>Lemma</u> 1** *: A pair of linear matrix equations*

$$|\psi_A\rangle' = \mathbf{m}_A\,|\psi_A\rangle\,, \ |\psi_B\rangle' = \mathbf{m}_B\,|\psi_B\rangle\,, \tag{2.9}$$

has an integral of motion $\langle\psi_A|\,\mathbf{n}\,|\psi_B\rangle$*, if there exists a constant matrix* $\mathbf{n}$ *such that*

$$\mathbf{n}\mathbf{m}_B + \mathbf{m}^\dagger{}_A\mathbf{n} = 0, \tag{2.10}$$

where the dagger denotes the adjoint matrix.

In 4WM the IOM are searched for in two possible forms, as $\langle\psi_i|\,\mathbf{n}\,|\psi_j\rangle$ or as $\langle\bar{\psi}_i|\,\mathbf{n}\,|\psi_j\rangle$. For the first form, we have the defining equation (2.10) specified as

$$\mathbf{n}\mathbf{m} + \mathbf{m}^\dagger\mathbf{n} = 0, \tag{2.11}$$

whereas for the second form of integrals, the defining equation is

$$\mathbf{n}\mathbf{m} + \mathbf{m}^T\mathbf{n} = 0. \tag{2.12}$$

For the general (complex) Γ, the unique solutions (up to rescaling by a constant) of these defining equations are the matrices

$$\mathbf{n}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} = \begin{cases} \mathbf{1} & \text{for TG,} \\ \sigma_3 & \text{for RG,} \end{cases} \tag{2.13}$$

for (2.11) and

$$\mathbf{n}_2 = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} = -\sigma i\sigma_2 = \begin{cases} -i\sigma_2 & \text{for TG,} \\ i\sigma_2 & \text{for RG,} \end{cases} \tag{2.14}$$

for (2.12), where σ_j are the Pauli matrices. The corresponding integrals are

$$\begin{array}{rcl} q_1 &=& \langle\psi_1|\,\mathbf{n}_1\,|\psi_1\rangle = I_1 + \sigma I_3, \\ q_2 &=& \langle\psi_2|\,\mathbf{n}_1\,|\psi_2\rangle = I_2 + \sigma I_4, \\ q_3 &=& \langle\psi_2|\,\mathbf{n}_1\,|\psi_1\rangle = B_1\bar{B}_4 - B_3\bar{B}_2, \\ q_4 &=& \langle\bar{\psi}_1|\,\mathbf{n}_2\,|\psi_2\rangle = B_1B_2 + \sigma B_3B_4. \end{array} \tag{2.15}$$

Since these IOM are present for arbitrary complex coupling Γ, they are said to be the *regular* IOM of the 4WM system. Later it will be shown that for special choices of Γ this system possesses additional (*exceptional*) IOM.

Not all of the conserved quantities q_1, q_2 , q_3, $\bar{q}_3$, q_4 and $\bar{q}_4$ are independent. There exists a relation

$$|q_4|^2 + \sigma|q_3|^2 = q_1q_2 \tag{2.16}$$

that reduces the number of (real) integrals of motion to five. Using the integrals, one can express the conjugated fields as dependent variables:

$$\begin{array}{rcl} \bar{B}_1 &=& [\sigma B_3\bar{q}_3 + B_2q_1]/q_4, \\ \bar{B}_2 &=& [B_1q_2 - \sigma B_4q_3]/q_4, \\ \bar{B}_3 &=& [B_4q_1 - B_1\bar{q}_3]/q_4, \\ \bar{B}_4 &=& [B_2q_3 + B_3q_2]/q_4. \end{array} \tag{2.17}$$

A more natural way to reduce the number of variables using conserved quantities is to introduce *polar* coordinates, suggested by the form of the conserved quantities q_1 and q_2:

$$\begin{array}{l} B_1 = \sqrt{q_1}c(\sigma,\alpha_1)\exp(i\beta_1), \\ B_2 = \sqrt{q_2}c(\sigma,\alpha_2)\exp(i\beta_2), \\ B_3 = \sqrt{q_1}s(\sigma,\alpha_1)\exp(i\gamma_1), \\ B_4 = \sqrt{q_2}s(\sigma,\alpha_2)\exp(i\gamma_2). \end{array} \tag{2.18}$$

Here the new variables are the six angles: α_1, α_2, β_1, β_2, γ_1, and γ_2. The symbols c and s stand for the trigonometric cosine and sine functions in the TG case, and for the hyperbolic cosine and sine in RG case (see Appendix A).

To further ilucidate the connection between IOM and the new variables, it is convenient to employ the integral q, introduced below [Eq. (2.41)]. It can be expressed in terms of the new variables, and the result is

$$\begin{aligned} q &= q_1^2 + q_2^2 + 2q_1q_2[c(\sigma,2\alpha_1)c(\sigma,2\alpha_2)+ \\ &+\sigma s(\sigma,2\alpha_1)s(\sigma,2\alpha_2)\cos(\Phi)], \end{aligned} \tag{2.19}$$

where $\Phi = \beta_1+\beta_2-\gamma_1-\gamma_2$ is the so-called relative phase. From the spherical and the hyperbolic [8] trigonometry it is known that the expression in brackets can be understood as a cosine (hyperbolic cosine) of some angle ρ, so that $2\alpha_1$, $2\alpha_2$ and ρ are the sides of a spherical (hyperbolic) triangle, and Φ is its central angle. Therefore

$$c(\sigma,\rho) = \frac{q - q_1^2 - q_2^2}{2q_1q_2} = const. \tag{2.20}$$

and Φ only depends on α_1 and α_2 :

$$\cos(\Phi) = \frac{c(\sigma,\rho) - c(\sigma,2\alpha_1)c(\sigma,2\alpha_2)}{\sigma s(\sigma,2\alpha_1)s(\sigma,2\alpha_2)}. \tag{2.21}$$

It is easy to check that:

$$\begin{aligned} |q_4|^2 &= q_1q_2c\left(\sigma,\frac{\rho}{2}\right)^2, \\ |q_3|^2 &= q_1q_2s\left(\sigma,\frac{\rho}{2}\right)^2, \end{aligned} \tag{2.22}$$

in agreement with Eq. (2.16). Thus, there are five independent real conserved quantities: q_1, q_2, ρ, and the phases of q_3 and q_4. The solution of EOM using these quantities is performed in Sec. IV.

C. Lax Pairs

The Lax pair representation (if it exists) helps determination of the integrals of motion. In general, if given dynamical system admits a Lax pair representation

$$\frac{d\hat{\mathcal{L}}}{dz} = [\hat{\mathcal{M}},\hat{\mathcal{L}}], \tag{2.23}$$

where $\hat{\mathcal{L}}$ and $\hat{\mathcal{M}}$ are suitably chosen operators or matrices, then all the traces $Tr(\hat{\mathcal{L}}^k)$ ($k \in \mathbb{N}$) are IOM. The determination of such a Lax pair of operators $(\hat{\mathcal{M}},\hat{\mathcal{L}})$ is usually the hardest part of the problem. The brackets in Eq. (2.23) stand for the *commutator*.

In the case of 4WM, the suitable matrices are easy to find, starting from the compact form of the spinorial EOM (2.7) and their conjugated equations:

$$\begin{aligned} \partial_z|\psi_\mu\rangle &= \mathbf{m}|\psi_\mu\rangle, \\ \partial_z\langle\psi_\mu| &= -\sigma\langle\psi_\mu|\mathbf{m}, \end{aligned} \tag{2.24}$$

where the index $\mu = i,\bar{i}$ and $|\psi_{\bar{i}}\rangle :\equiv \mathbf{n}_1\mathbf{n}_2|\bar{\psi}_i\rangle$. The following matrices

$$\mathcal{L}_{\mu\nu} :\equiv |\psi_\mu\rangle\langle\psi_\nu|\mathbf{n}_1, \tag{2.25}$$

satisfy the Lax pair equations

$$\partial_z\mathcal{L}_{\mu\nu} = [\mathbf{m},\mathcal{L}_{\mu\nu}]. \tag{2.26}$$

The corresponding Laxian IOM are given by

$$q_{\mu_1\nu_1\cdots\mu_k\nu_k} :\equiv \mathrm{Tr}\left(\mathcal{L}_{\mu_1\nu_1}\cdots\mathcal{L}_{\mu_k\nu_k}\right). \tag{2.27}$$

For $k=1$ we have

$$\left(q_{(1)\mu\nu}\right) = \begin{pmatrix} q_1 & q_3 & 0 & -q_4 \\ \bar{q}_3 & q_2 & -q_4 & 0 \\ 0 & -\bar{q}_4 & \sigma q_1 & \sigma\bar{q}_3 \\ -\bar{q}_4 & 0 & \sigma q_3 & q_2 \end{pmatrix}. \tag{2.28}$$

For higher k, the resulting IOM are the products of $q_{(1)}$. For example,

$$q_{(2)\mu\nu\alpha\beta} = q_{(1)\alpha\nu}q_{(1)\mu\beta}.$$

Thus, the higher Laxian IOM are not yielding any new independent integrals.

An alternative variant of the Lax pair approach is presented in Appendix B.

D. I-Symmetry

• <u>**Definition**</u> **1** : *The symmetry of the set of integrals* $\{q_\alpha\}$ *(***the I-symmetry***) is the mapping* $\{x^\mu\} \to \{x'^\mu\}$ *which preserves the analytical structure* $(x')^{i+4} = ((x')^i)^*$ *and leaves all the integrals invariant* $q_\alpha(x') = q_\alpha(x)$.

I-symmetries define the algebraic structure of the sistem at hand. In practice one first calculates the *infinitesimal* I-symmetries, given by

$$\delta q_\alpha(x) \equiv \vec{l}q_\alpha \equiv \delta x^\mu\partial_\mu q_\alpha = 0, \tag{2.29}$$

where $\vec{l} = \omega^\mu(x)\partial_\mu$ is the generating vector field, and then establishes the *large* (non-infinitesimal) I-symmetries, by exponentiating the infinitesimal ones. This is the standard procedure in the theory of Lie-groups.

Although in general the coefficients ω^μ are nonlinear functions of $\{x\}$ (the nonlinear I-symmetries), here only the linear I-symmetry algebras will be considered. These can be calculated easily by a linear ansatz $\omega^\mu = a^\mu{}_\nu x^\nu$. In this way a general linear symmetry of the full set $\{q_1,\cdots,\bar{q}_4\}$ of the regular IOM is found:

$$\begin{aligned} 2\delta B_1 &= +i\epsilon_3B_1 + (\epsilon_2+i\epsilon_1)B_3, \\ 2\delta B_2 &= -i\epsilon_3B_2 + (\epsilon_2-i\epsilon_1)B_4, \\ 2\delta B_3 &= -i\epsilon_3B_3 - \sigma(\epsilon_2-i\epsilon_1)B_1, \\ 2\delta B_4 &= +i\epsilon_3B_4 - \sigma(\epsilon_2+i\epsilon_1)B_2. \end{aligned} \tag{2.30}$$

In the spinor notation the matrix form of I-symmetries is

$$\delta |\psi_{1,2}\rangle = \mathbf{\Sigma}^T |\psi_{1,2}\rangle \,, \tag{2.31}$$

where the traceless matrix $\mathbf{\Sigma}$ is given by

$$\mathbf{\Sigma} = \frac{1}{2}\begin{pmatrix} i\epsilon_3 & -\sigma(\epsilon_2 - i\epsilon_1) \\ \epsilon_2 + i\epsilon_1 & -i\epsilon_3 \end{pmatrix} = i\epsilon_a \mathbf{S}_a, \tag{2.32}$$

and $\{\epsilon_a\}$ are real parameters. The basis matrices (the *generators*) are

$$\begin{aligned} \mathbf{S}_1 &= \frac{1}{2}\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix} = \begin{cases} \frac{1}{2}\sigma_1 & \text{for TG,} \\ -\frac{i}{2}\sigma_2 & \text{for RG,} \end{cases} \\ \mathbf{S}_2 &= \frac{1}{2}\begin{pmatrix} 0 & \sigma i \\ -i & 0 \end{pmatrix} = \begin{cases} -\frac{1}{2}\sigma_2 & \text{for TG,} \\ -\frac{i}{2}\sigma_1 & \text{for RG,} \end{cases} \\ \mathbf{S}_3 &= \frac{1}{2}\sigma_3. \end{aligned} \tag{2.33}$$

The set of all matrices $\mathbf{\Sigma}$ that are traceless and satisfy the generalized hermiticity condition $\mathbf{\Sigma}^\dagger \eta + \eta \mathbf{\Sigma} = 0$ forms the Lie-algebra $su(2)$ for TG, and $su(1,1)$ for RG. The matrix $\eta = \mathbf{n}_1$ is called the $su(2)/su(1,1)$ *metric* matrix. The generators $\{\mathbf{S}_a\}$ obey the standard commutation relations

$$\begin{aligned} [\mathbf{S}_1, \mathbf{S}_2] &= -i\sigma \mathbf{S}_3 \,, \\ [\mathbf{S}_2, \mathbf{S}_3] &= -i\mathbf{S}_1 \,, \\ [\mathbf{S}_3, \mathbf{S}_1] &= -i\mathbf{S}_2 \,. \end{aligned} \tag{2.34}$$

Thus, both $|\psi_1\rangle$ and $|\psi_2\rangle$ are transforming according to the *fundamental* (spinorial) representation of the corresponding algebra g_I.

Every finite Lie-algebra g has the corresponding Lie-group $\mathcal{G}$ of "large" transformations, obtained via exponential mapping:

$$\forall \mathbf{\Sigma} \in g \Rightarrow \mathbf{G} :\equiv \exp(i\mathbf{\Sigma}) \in \mathcal{G}.$$

For $g_I = su(2)$ (the TG case), the group is $\mathcal{G}_I = SU(2)$, and for $g_I = su(1,1)$ (the RG case), the group is $SU(1,1)$, the noncompact version of $SU(2)$. Both groups can be represented by sets of 2×2 complex matrices $\mathbf{G}$ that are unimodular ($\det \mathbf{G} = 1$) and (pseudo)unitary ($\mathbf{G}^\dagger \eta \mathbf{G} = \eta$).

The Cayley-Klein parameterization of the general $SU(2)/SU(1,1)$ group element

$$\mathbf{G} = \begin{pmatrix} y_1 + iy_2 & y_3 + iy_4 \\ -\sigma(y_3 - iy_4) & y_1 - iy_2 \end{pmatrix}, \tag{2.35}$$

where $y_{1,\cdots,4} \in \mathbb{R}$, turns the unimodality condition into a geometric relation (the definition of the parameter manifold of the group $\mathcal{G}_I$):

$$\det \mathbf{G} \;=\; (y_1)^2 + (y_2)^2 + \sigma\left[(y_3)^2 + (y_4)^2\right] = 1. \tag{2.36}$$

Thus, the parameter *manifold* for $SU(2)_I$ is the sphere $\mathbb{S}^3$, and for $SU(1,1)_I$ it is the hyperboloid $\mathbb{H}^3$, both embedded in $\mathbb{R}^4$. (For a short classification of hyperboloids in $\mathbb{R}^4$ see Appendix C.)

From this fact alone, one could expect that the TG case will be expressed in a natural way in terms of the trigonometric functions, and the RG case in terms of both the trigonometric functions (compact dimensions) and the hyperbolic functions (noncompact dimensions). In this sense the cases are "twins", i.e. there is a number of equations holding in both cases, up to the exchange of the trigonometric/hyperbolic functions.

E. Action of I-Symmetries on the Lax variables

It is of interest to know the action of the I-symmetries on the Lax matrices $\mathcal{L}_{\mu\nu}$. Since the Lax matrices are constructed out of the basic spinors, some regularity must be induced in the transformation law of these variables.

For example, for $\mathcal{L}_{11} = |\psi_1\rangle\langle\psi_1|\mathbf{n}_1$, the action of the infinitesimal I-symmetry yields

$$\delta\mathcal{L}_{11} = \mathbf{\Sigma}^T \mathcal{L}_{11} + \mathcal{L}_{11}\mathbf{n}_1^{-1}\mathbf{\Sigma}^*\mathbf{n}_1.$$

Owing to matrix identities $\mathbf{n}_1^{-1} = \mathbf{n}_1$ and $\mathbf{n}_1\mathbf{\Sigma}^*\mathbf{n}_1 = -\mathbf{\Sigma}^T$, this expression simplifies to

$$\delta\mathcal{L}_{11} = \left[\mathbf{\Sigma}^T, \mathcal{L}_{11}\right]. \tag{2.37}$$

This represents the *adjoint action* of the I-symmetry on $\mathcal{L}_{11}$. In a similar way, one finds that the same transformation law is valid for all $\mathcal{L}_{\mu\nu}$:

$$\delta\mathcal{L}_{\mu\nu} \;=\; \left[\mathbf{\Sigma}^T, \mathcal{L}_{\mu\nu}\right]. \tag{2.38}$$

Thus, due to cyclic invariance of the matrix trace operation, all Laxian IOM are invariant upon the action of I-symmetries. This is expected.

F. Exceptional IOM

The "regular" IOM, obtained in the subsection II B, form the full set of IOM for the complex coupling Γ. However, in the special cases when Γ is either real or imaginary, there exist additional IOM. These will be called the "exceptional" IOM.

To see the significance of these special cases, let us evaluate the "time"-change of the grating amplitude Q:

$$IQ' = -\Gamma Q q_5. \tag{2.39}$$

Here q_5 is the expression $q_5 = I_1 + I_2 - \sigma(I_3 + I_4)$, whose "time"-change is

$$Iq_5' = 4\sigma \mathrm{Re}\,(\Gamma)|Q|^2. \tag{2.40}$$

Notice that q_5 is IOM in the case of imaginary Γ (so, it is an "exceptional" IOM). However, when Γ is a complex number, this quantity turns out to be a suitable variable for later calculations.

From equations (2.39) and (2.40) another conserved quantity (for the general, complex Γ) is obtained:

$$q = q_5^2 + 4\sigma|Q|^2. \tag{2.41}$$

This quantity is IOM of the fourth order. It does not carry any independent information, since it is reducible to the already known regular IOM:

$$q = (q_1 + q_2)^2 - 4\sigma|q_3|^2. \tag{2.42}$$

Nevertheless, q_5 plays an important role in one of the two presented procedures for solving EOM in the $\Gamma \in \mathbb{R}$ case.

From Eq. (2.39), two important relations follow:

$$\begin{aligned} I|Q|' &= -\mathrm{Re}\ (\Gamma)|Q|q_5, \\ I\arg(Q)' &= -\mathrm{Im}\ (\Gamma)q_5. \end{aligned} \tag{2.43}$$

These equations indicate the existence of two important special cases: $\Gamma \in \mathbb{R}$ and $\Gamma \in i\mathbb{R}$. The case $\mathrm{Im}\ \Gamma = 0$ implies $\arg Q = const$, so that $\phi :\equiv \arg \mu = const$, while the case $\mathrm{Re}\ \Gamma = 0$ implies that $|Q| = const$. The $\Gamma \in \mathbb{R}$ case is considered first.

1. $\Gamma \in \mathbb{R}$

Analysis of this case is based on the fact that the phase ϕ of the grating amplitude Q is constant for real couplings. This allows introduction of a new independent variable $\theta(z) = \int_0^z dz'|\mu(z')| + \theta_0$, which casts the problem into a linear form. The matrix $\mathbf{m}$ is now replaced by

$$\tilde{\mathbf{m}} = \begin{pmatrix} 0 & \sigma\nu \\ -\bar{\nu} & 0 \end{pmatrix}, \tag{2.44}$$

where $\nu :\equiv \exp(i\phi) = const$.

The defining relation (2.11) has as solutions not only the matrix $\mathbf{n}_1$, but also the new one:

$$\mathbf{n}_3 \equiv \begin{pmatrix} 0 & \nu \\ -\bar{\nu} & 0 \end{pmatrix}, \tag{2.45}$$

which is anti-Hermitian $\mathbf{n}_3^\dagger = -\mathbf{n}_3$. Along the same lines, the defining relation (2.12) has as solutions both the matrix $\mathbf{n}_2$ and the new one:

$$\mathbf{n}_4 \equiv \begin{pmatrix} \bar{\nu} & 0 \\ 0 & \sigma\nu \end{pmatrix}, \tag{2.46}$$

which is symmetric. Having the new matrices $\mathbf{n}_3$ and $\mathbf{n}_4$ that satisfy the Lemma, a set of additional conserved quantities can be constructed:

$$\begin{aligned} w_1 &:\equiv \langle\psi_1|\,\mathbf{n}_3\,|\psi_1\rangle = 2i\mathrm{Im}\ (\nu\bar{B}_1B_3) \\ w_2 &:\equiv \langle\psi_2|\,\mathbf{n}_3\,|\psi_2\rangle = -2i\sigma\mathrm{Im}\ (\nu\bar{B}_4B_2), \\ w_3 &:\equiv \langle\psi_1|\,\mathbf{n}_3\,|\psi_2\rangle = -\sigma\nu\bar{B}_1B_2 - \bar{\nu}\bar{B}_3B_4 \\ w_4 &:\equiv \langle\bar{\psi}_1|\,\mathbf{n}_4\,|\psi_1\rangle = \bar{\nu}B_1^2 + \sigma\nu B_3^2, \\ w_5 &:\equiv \langle\bar{\psi}_1|\,\mathbf{n}_4\,|\psi_2\rangle = \bar{\nu}B_1B_4 - \nu B_3B_2, \\ w_6 &:\equiv \langle\bar{\psi}_2|\,\mathbf{n}_4\,|\psi_2\rangle = \bar{\nu}B_4^2 + \sigma\nu B_2^2. \end{aligned} \tag{2.47}$$

Note that $\langle\psi_2|\,\mathbf{n}_3\,|\psi_1\rangle = -w_3$ and $\langle\bar{\psi}_2|\ \mathbf{n}_4\ |\psi_1\rangle = w_5$.

One can extend the Lax procedure, in the spirit of the subsection II C, to this case as well. The Lax matrices are now $\mathcal{L}_{\mu\nu}^{(\mathbb{R})} :\equiv |\psi_\mu\rangle\langle\psi_\nu|\,\mathbf{n}_3$, and the corresponding Lax equations

$$\partial_\theta\mathcal{L}_{\mu\nu}^{(\mathbb{R})} = [\tilde{\mathbf{m}}, \mathcal{L}_{\mu\nu}^{(\mathbb{R})}]. \tag{2.48}$$

The corresponding Laxian IOM of the first order are

$$\mathrm{Tr}\mathcal{L}_{\mu\nu}^{(\mathbb{R})} = \begin{pmatrix} w_1 & -\bar{w}_3 & w_4 & w_5 \\ w_3 & w_2 & w_5 & w_6 \\ -\bar{w}_4 & -\bar{w}_5 & -\sigma w_1 & -\sigma w_3 \\ -\bar{w}_5 & -\bar{w}_6 & \sigma\bar{w}_3 & -\sigma w_2 \end{pmatrix}. \tag{2.49}$$

These IOM are the same as the ones already obtained through the spinorial approach. As mentioned, a more general Lax pair procedure is presented in Appendix B.

Action of an I-symmetry on $\mathcal{L}_{\mu\nu}^{(\mathbb{R})}$ produces

$$\delta\mathcal{L}_{\mu\nu}^{(\mathbb{R})} = \mathbf{\Sigma}^T\mathcal{L}_{\mu\nu}^{(\mathbb{R})} + \mathcal{L}_{\mu\nu}^{(\mathbb{R})}\mathbf{n}_3^{-1}\mathbf{\Sigma}^{T\dagger}\mathbf{n}_3.$$

Here the condition $\mathbf{n}_3^{-1}\mathbf{\Sigma}^{T\dagger}\mathbf{n}_3 = -\mathbf{\Sigma}^T$, necessary for the covariant form of action, can be achieved in different ways:

• Case 1: $\nu^2 = -\sigma$. This corresponds to $\phi = \pm\pi/2$ (in TG) and to $\phi = 0$ or π (in RG). This case allows for the full su_I symmetry, i.e. all three ϵ_a parameters can have non-zero values. However, only the diagonal part of $\mathbf{\Sigma}^T$ figures in the transformation law:

$$\delta\mathcal{L}_{\mu\nu}^{(\mathbb{R})} = \frac{i\epsilon_3}{2}[\sigma_3, \mathcal{L}_{\mu\nu}^{(\mathbb{R})}]. \tag{2.50}$$

• Case 2: $\nu^2 \neq -\sigma$. Here only the diagonal part of I-symmetries survives, i.e. ϵ_1 and ϵ_2 have to be set equal to zero. The transformation law still has the same form as in the case above.

Thus, the w IOM are invariant under the full su_I symmetry algebra if $\nu^2 = -\sigma$, and under the u_1 subalgebra generated by $\sigma_3/2$ if $\nu^2 \neq -\sigma$.

An important special case is the phase conjugation, when the relative phase Φ ($\equiv \beta_1 + \beta_2 - \gamma_1 - \gamma_2$) is constant (0 or π). Then, using relations (2.18) and the fact that the argument ϕ ($= \beta_1 - \gamma_1 = \gamma_2 - \beta_2$) of μ is constant, the following values for the integrals w are obtained:

$$\begin{aligned} w_1 &= 0, \\ w_2 &= 0, \\ w_3 &= -\sqrt{q_1q_2}c(\sigma, \alpha_1 - \alpha_2)\exp(i(\beta_2 - \gamma_1)), \\ w_4 &= q_1\exp(i(\beta_1 + \gamma_1)), \\ w_5 &= \sigma q_2\exp(i(\beta_2 + \gamma_2)), \\ w_6 &= -\sqrt{q_1q_2}s(\sigma, \alpha_1 - \alpha_2)\exp(i(\beta_1 + \beta_2)). \end{aligned} \tag{2.51}$$

These relations imply that all the phases β_1, β_2, γ_1, γ_2 are constant, and that the α-variables are linearly dependent: $\alpha_1 - \alpha_2 =$ constant. Hence, all the fields essentially depend on only one real quantity, for example on α_1.

2. $\Gamma \in i\mathbb{R}$

The case of Γ imaginary has only one exceptional integral, q_5. The corresponding I-symmetry of the set $\{q_1, ..., q_5\}$ is restricted to the diagonal part of Eq. (2.30):

$$
\begin{aligned}
2\delta B_1 &= i\epsilon_3 B_1, \\
2\delta B_2 &= -i\epsilon_3 B_2, \\
2\delta B_3 &= -i\epsilon_3 B_3, \\
2\delta B_4 &= i\epsilon_3 B_4 \ .
\end{aligned}
\tag{2.52}
$$

This is a $u(1)$ algebra of the transformations:

$$
\delta \left|\psi_{1,2}\right\rangle = i\frac{\epsilon_3}{2}\sigma_3 \left|\psi_{1,2}\right\rangle , \tag{2.53}
$$

and the corresponding group is $U(1)$. The parameter space of this group is the circle $\mathbb{S}^1$ of circumference 4π. That group is the subgroup of both $SU(2)_I$ and $SU(1,1)_I$ groups.

III. SYMMETRIES OF THE EQUATIONS OF MOTION

In general, one should distinguish the symmetries of the integrals of motion from the symmetries of the equations of motion.

• **Definition 2** *:* **E-symmetries** *[9]: Any vector-field $\vec{L}$ that satisfies the master equation*

$$
\left[\vec{L}, \vec{F}\right] = 0, \tag{3.1}
$$

is the symmetry of the dynamical equations (1.4).

The set g_E of E-symmetries is also a Lie-algebra, i.e. it is linear, and the commutator of any two E-symmetries is another E-symmetry. So, one can describe the full algebra by its generators and their commutation relations.

The E-symmetries are sought in the form of the most general linear ansatz

$$
\vec{L} = x^\mu a_\mu{}^\nu \partial_\nu . \tag{3.2}
$$

After some algebra, six generators are found for the 4WM system:

$$
\begin{aligned}
\vec{L}_0 &= B_1\partial_1 + B_2\partial_2 + B_3\partial_3 + B_4\partial_4 + c.c., \\
\vec{L}_1 &= i(B_1\partial_1 + B_2\partial_2 + B_3\partial_3 + B_4\partial_4 - c.c.), \\
\vec{L}_2 &= i(B_1\partial_1 - B_2\partial_2 - c.c.), \\
\vec{L}_3 &= i(B_3\partial_3 - B_4\partial_4 - c.c.), \\
\vec{L}_4 &= \bar{B}_2\partial_1 - \bar{B}_1\partial_2 + \bar{B}_4\partial_3 - \bar{B}_3\partial_4 + c.c., \\
\vec{L}_5 &= i(\bar{B}_2\partial_1 - \bar{B}_1\partial_2 + \bar{B}_4\partial_3 - \bar{B}_3\partial_4 - c.c.).
\end{aligned}
\tag{3.3}
$$

These six generators form the complete set of linear E-symmetries for the general (complex) Γ. They generate the algebra $g_E \equiv r \oplus u(1) \oplus u(1) \oplus su(1,1)$, with the non-vanishing commutators

$$
\begin{aligned}
[\vec{L}_1, \vec{L}_4] &= -2\vec{L}_5, \\
[\vec{L}_1, \vec{L}_5] &= 2\vec{L}_4, \\
[\vec{L}_4, \vec{L}_5] &= 2\vec{L}_1.
\end{aligned}
\tag{3.4}
$$

Defining the general infinitesimal E-symmetry by $\mathfrak{d} \equiv \sum_{i=0}^{5} \theta_i \vec{L}_i$, we find the transformation law of the fields:

$$
\begin{aligned}
\mathfrak{d}B_1 &= [\theta_0 + i(\theta_1 + \theta_2)]B_1 + (\theta_4 + i\theta_5)\bar{B}_2, \\
\mathfrak{d}B_2 &= [\theta_0 + i(\theta_1 - \theta_2)]B_2 - (\theta_4 + i\theta_5)\bar{B}_1, \\
\mathfrak{d}B_3 &= [\theta_0 + i(\theta_1 + \theta_3)]B_3 + (\theta_4 + i\theta_5)\bar{B}_4, \\
\mathfrak{d}B_4 &= [\theta_0 + i(\theta_1 - \theta_3)]B_4 - (\theta_4 + i\theta_5)\bar{B}_3,
\end{aligned}
\tag{3.5}
$$

or, in a more compact notation:

$$
\begin{aligned}
\mathfrak{d}|\psi_1\rangle &= \left[(\theta_0 + i\theta_1)\mathbf{1} + i\begin{pmatrix}\theta_2 & 0 \\ 0 & \theta_3\end{pmatrix}\right]|\psi_1\rangle + \\
&\quad -(\theta_4 + i\theta_5)|\psi_{\bar{2}}\rangle, \\
\mathfrak{d}|\psi_2\rangle &= \left[(\theta_0 + i\theta_1)\mathbf{1} - i\begin{pmatrix}\theta_2 & 0 \\ 0 & \theta_3\end{pmatrix}\right]|\psi_2\rangle + \\
&\quad -\sigma(\theta_4 + i\theta_5)|\psi_{\bar{1}}\rangle.
\end{aligned}
\tag{3.6}
$$

The parameters θ_0 and θ_4 correspond to the two noncompact dimensions of the symmetry group $\mathcal{G}_E$, i.e. their values are arbitrary real numbers. This is in contrast to the rest of the parameters, which are periodic. So, the group of E-symmetries $\mathcal{G}_E \equiv \exp g_E$ is $\mathbb{R} \otimes U(1)^2 \otimes SU(1,1)$.

It is easy to check [from the defining relation (3.1)] that the E-symmetries always map the integrals of motion to the integrals of motion (and also the solutions of EOM to the solutions of EOM). Hence:

$$
\begin{aligned}
\mathfrak{d}q_1 &= 2\theta_0 q_1 + (\theta_4 + i\theta_5)\bar{q}_4 + (\theta_4 - i\theta_5)q_4, \\
\mathfrak{d}q_2 &= 2\theta_0 q_2 - (\theta_4 + i\theta_5)\bar{q}_4 - (\theta_4 - i\theta_5)q_4, \\
\mathfrak{d}q_3 &= [2\theta_0 + i(\theta_2 + \theta_3)]q_3, \\
\mathfrak{d}q_4 &= 2(\theta_0 + i\theta_1)q_4 + (\theta_4 + i\theta_5)(q_2 - q_1).
\end{aligned}
\tag{3.7}
$$

This is a version of the Nöther theorem: If one of IOM is taken as the "Hamiltonian" H of the system, then the action $\vec{L}_i(H)$ of each $\vec{L}_i$ on such a Hamiltonian produces another IOM.

Here, the following linear combinations of regular IOM are forming the irreducible representations of the $u(1)_{L_0} \oplus su(1,1)_{L_1,L_4,L_5}$ algebra under the E-symmetries: $q_1 + q_2$, q_3 and $\bar{q}_3$ form singlets

$$
\begin{aligned}
\mathfrak{d}(q_1 + q_2) &= 2\theta_0(q_1 + q_2), \\
\mathfrak{d}q_3 &= [2\theta_0 + i(\theta_2 + \theta_3)]q_3, \\
\mathfrak{d}\bar{q}_3 &= [2\theta_0 - i(\theta_2 + \theta_3)]\bar{q}_3,
\end{aligned}
\tag{3.8}
$$

while $|T\rangle :\equiv \left\{q_4, (q_1 - q_2)/\sqrt{2}, \bar{q}_4\right\}^T$ is transforming as the triplet representation $\mathfrak{d}|T\rangle = \mathbf{P}|T\rangle$, where $\mathbf{P}$ is given by

$$
\begin{pmatrix}
2(\theta_0 + i\theta_1) & -\sqrt{2}(\theta_4 + i\theta_5) & 0 \\
\sqrt{2}(\theta_4 - i\theta_5) & 2\theta_0 & \sqrt{2}(\theta_4 + i\theta_5) \\
0 & -\sqrt{2}(\theta_4 - i\theta_5) & 2(\theta_0 - i\theta_1)
\end{pmatrix}.
$$

Hence, one can start from the knowledge of only q_1 and recover the (almost) full set of regular integrals $\{q_1, q_2, q_4, \bar{q}_4\}$, by acting on them with the E-symmetries.

A. $\Gamma\in\mathbb{R}$

In the $\Gamma\in\mathbb{R}$ case, one can apply the same linear ansatz as in the general case. The set of linear E-symmetries thus obtained is $\{\vec{L}_0,\cdots,\vec{L}_7\}$, where $\vec{L}_{0-5}$ are the already known generators (3.3), and the two additional generators are found for the RG case:

$$\begin{aligned}\vec{L}_6 &= B_4\partial_1 - B_3\partial_2 + B_2\partial_3 - B_1\partial_4 + c.c.,\\ \vec{L}_7 &= i(\bar{B}_4\partial_1 + \bar{B}_3\partial_2 + \bar{B}_2\partial_3 + \bar{B}_1\partial_4 - c.c.).\end{aligned} \tag{3.9}$$

An alternative approach is to perform the redefinition $z\to\theta(z)$ of the "time" variable (introduced in the previous section), making the matrix $\tilde{\mathbf{m}}=\begin{pmatrix}0 & \sigma\nu\\ -\bar{\nu} & 0\end{pmatrix}$ constant. This allows one to translate the vector-field language (applicable in the general case) into the matrix language. Define the column

$$|\Psi\rangle\equiv\begin{pmatrix}|\psi_1\rangle\\ |\psi_2\rangle\\ |\psi_{\bar{1}}\rangle\\ |\psi_{\bar{2}}\rangle\end{pmatrix}. \tag{3.10}$$

The evolution equation (2.7) can now be written as

$$\partial_\theta|\Psi\rangle=\mathbf{M}|\Psi\rangle, \tag{3.11}$$

where the constant evolution matrix is

$$\mathbf{M}:\equiv\mathbf{1}_4\otimes\tilde{\mathbf{m}}=\begin{pmatrix}\tilde{\mathbf{m}} & 0 & 0 & 0\\ 0 & \tilde{\mathbf{m}} & 0 & 0\\ 0 & 0 & \tilde{\mathbf{m}} & 0\\ 0 & 0 & 0 & \tilde{\mathbf{m}}\end{pmatrix}. \tag{3.12}$$

The master equation (3.1) now has the matrix form

$$[\mathbf{K},\mathbf{M}]=0, \tag{3.13}$$

where the matrix $\mathbf{K}=(\mathbf{K}_{\mu\nu})$ defines the infinitesimal symmetry of the big "spinor" $\mathfrak{d}|\Psi\rangle=\mathbf{K}|\Psi\rangle$ (here $\mu,\nu\in\{1,2,\bar{1},\bar{2}\}$). The above master equation is translated into "smaller" versions $[\mathbf{K}_{\mu\nu},\tilde{\mathbf{m}}]=0$, valid for each 2×2 block matrix $\mathbf{K}_{\mu\nu}$. Solutions of these "small" master equations are all of the same form

$$\mathbf{K}_{\mu\nu}=\alpha_{\mu\nu}\mathbf{1}_2+\beta_{\mu\nu}\tilde{\mathbf{m}},\qquad(\forall\mu,\nu). \tag{3.14}$$

The analiticity conditions $|\psi_{\bar{i}}\rangle=\mathbf{n}_1\mathbf{n}_2\,|\bar{\psi}_i\rangle$ yield the constraints

$$\begin{aligned}\alpha_{\bar{i}j}=-\sigma\overline{\alpha_{i\bar{j}}},&\quad \beta_{\bar{i}j}=-\sigma\overline{\beta_{i\bar{j}}},\\ \alpha_{\bar{i}\bar{j}}=\overline{\alpha_{ij}},&\quad \beta_{\bar{i}\bar{j}}=\overline{\beta_{ij}},\end{aligned} \tag{3.15}$$

i.e.

$$\begin{aligned}\mathbf{K}_{\bar{i}j} &= -\sigma\overline{\alpha_{i\bar{j}}}\mathbf{1}_2-\sigma\overline{\beta_{i\bar{j}}}\tilde{\mathbf{m}},\\ \mathbf{K}_{\bar{i}\bar{j}} &= \overline{\alpha_{\bar{i}\bar{j}}}\mathbf{1}_2+\overline{\beta_{\bar{i}\bar{j}}}\tilde{\mathbf{m}}.\end{aligned} \tag{3.16}$$

Hence:

$$\begin{aligned}\mathfrak{d}\,|\psi_i\rangle &= \alpha_{ij}\,|\psi_j\rangle+\beta_{ij}\tilde{\mathbf{m}}\,|\psi_j\rangle+\\ &\quad+\alpha_{i\bar{j}}\,|\psi_{\bar{j}}\rangle+\beta_{i\bar{j}}\tilde{\mathbf{m}}\,|\psi_{\bar{j}}\rangle,\\ \mathfrak{d}\,|\psi_{\bar{i}}\rangle &= -\sigma\overline{\alpha_{i\bar{j}}}\,|\psi_j\rangle-\sigma\overline{\beta_{i\bar{j}}}\tilde{\mathbf{m}}\,|\psi_j\rangle+\\ &\quad+\overline{\alpha_{ij}}\,|\psi_{\bar{j}}\rangle+\overline{\beta_{ij}}\tilde{\mathbf{m}}\,|\psi_{\bar{j}}\rangle.\end{aligned} \tag{3.17}$$

In this way the rescaled EOM have 32 symmetries, characterized by the the real and the imaginary parts of the parameters $\{\alpha_{ij},\beta_{ij},\alpha_{i\bar{j}},\beta_{i\bar{j}}\}$ $(i,j=1,2)$.

B. On the relation between I-symmetries and E-symmetries

One may ask the question, what is the relation between the two groups of symmetries: I-symmetries and E-symmetries? The following general consideration clarifies this issue a bit. Let $\vec{F}$ be the EOM vector field, δ an arbitrary E-symmetry, $\mathfrak{d}$ an arbitrary I-symmetry, and q an arbitrary IOM. Since $[\vec{F},\mathfrak{d}]\equiv0$ and $\vec{F}(q)\equiv0$, it follows that $\vec{F}(\mathfrak{d}q)=0$, i.e. $\mathfrak{d}q\sim q$ (Nöther theorem: E-symmetry of IOM is also IOM). From this conclusion and from $\delta q\equiv0$ it follows that $[\mathfrak{d},\delta]q=0$, i.e. $[\mathfrak{d},\delta]\sim\delta$ (E-symmetry maps an I-symmetry into another I-symmetry). Hence, one expects that $[\vec{L}_i,\vec{S}_a]\sim\vec{S}_b$.

This can be explicitly checked in the case of 4WM system: the generators $\vec{L}_i$ for $i\in\{0,1,4,5\}$ commute with all three $\vec{S}_a$ generators, whereas the remaining two E-symmetries $\vec{L}_{2,3}$ have nontrivial commutators with $\vec{S}_a$:

$$\begin{aligned}[][\vec{L}_2,\vec{S}_1\pm\vec{S}_2] &= \mp i\left(\vec{S}_1\pm\vec{S}_2\right),\\ [\vec{L}_3,\vec{S}_1\pm\vec{S}_2] &= \pm i\left(\vec{S}_1\pm\vec{S}_2\right),\\ [\vec{L}_{2,3},\vec{S}_3] &= 0.\end{aligned} \tag{3.18}$$

Thus the $su(1,1)_E$ symmetry (generated by $\{\vec{L}_1,\vec{L}_4,\vec{L}_5\}$) commutes with the $su(2)_I/su(1,1)_I$ symmetry.

IV. SOLUTION PROCEDURES

A. $\Gamma\in\mathbb{R}$: The first procedure

We present in detail the solution procedures for the case when Γ is real. This case is physically the most relevant. The case when Γ is imaginary, is treated similarly.

The equations for α_1 and α_2, extracted from Eqs. (1.4), form a closed system of equations:

$$\begin{aligned}2I\alpha_1' &= -\Gamma\left[q_1 s(\sigma,2\alpha_1)+q_2 s(\sigma,2\alpha_2)\cos(\Phi)\right],\\ 2I\alpha_2' &= -\Gamma\left[q_1 s(\sigma,2\alpha_1)\cos(\Phi)+q_2 s(\sigma,2\alpha_2)\right],\end{aligned} \tag{4.1}$$

which can be integrated with little difficulty. Once α_1 and α_2 are known, the remaining four angles are found easily:

$$2I\beta_1' = -\sigma\Gamma q_2\sin(\Phi)s(\sigma,2\alpha_2)t(\sigma,\alpha_1),$$

$$
\begin{aligned}
2I\beta_2' &= -\sigma\Gamma q_1 \sin(\Phi) s(\sigma, 2\alpha_1) t(\sigma, \alpha_2), \\
2I\gamma_1' &= -\Gamma q_2 \sin(\Phi) s(\sigma, 2\alpha_2) ct(\sigma, \alpha_1), \\
2I\gamma_2' &= -\Gamma q_1 \sin(\Phi) s(\sigma, 2\alpha_1) ct(\sigma, \alpha_2),
\end{aligned} \tag{4.2}
$$

where t and ct are the remaining two trigonometric/hyperbolic functions, formed by using the rule (A4) (see Appendix A).

Equations (4.1) are integrated as follows. First, two new variables are introduced:

$$
x_1 = c(\sigma, 2\alpha_1), \quad x_2 = c(\sigma, 2\alpha_2). \tag{4.3}
$$

In terms of these variables $q_5 = q_1x_1 + q_2x_2$, and Eqs. (4.1) become

$$
\begin{aligned}
Ix_1' &= \Gamma\left[q_1 + q_2 c(\sigma, \rho) - x_1 q_5\right], \\
Ix_2' &= \Gamma\left[q_2 + q_1 c(\sigma, \rho) - x_2 q_5\right].
\end{aligned} \tag{4.4}
$$

Note that, due to symmetry, only one of Eqs. (4.4) is independent. The solution of the other is obtained from the solution of the first one by interchanging q_1 and q_2 . This, however, holds only when Γ is real. On the other hand, using Eq. (2.41), Eq. (2.40) can be written as:

$$
Iq_5' = \Gamma\left(q - q_5^2\right). \tag{4.5}
$$

The integration of this equation depends on the geometry. For TG, the total intensity is constant ($I = q_1 + q_2$), so that

$$
\int \frac{dq_5}{q - q_5^2} = \frac{\Gamma z}{I}. \tag{4.6}
$$

For RG, the intensity is not constant, and

$$
\int \frac{q_5 dq_5}{q - q_5^2} = \Gamma z. \tag{4.7}
$$

The value of the integral in TG depends on whether q is larger or smaller than q_5^2 . For the first case:

$$
q_5(z) = \sqrt{q}\tanh\left[\tanh^{-1}\left(\frac{q_5(0)}{\sqrt{q}}\right) + \frac{\sqrt{q}}{I}\Gamma z\right] \tag{4.8}
$$

whereas for the second:

$$
q_5(z) = \sqrt{q}\,\text{cotanh}\left[\text{cotanh}^{-1}\left(\frac{q_5(0)}{\sqrt{q}}\right) + \frac{\sqrt{q}}{I}\Gamma z\right]. \tag{4.9}
$$

In RG:

$$
q_5^2(z) = q_5^2(0)\exp(-2\Gamma z) + q\left[1 - \exp(-2\Gamma z)\right]. \tag{4.10}
$$

Once q_5 is determined, Eqs. (4.4) for x_1 and x_2 (i.e. α_1 and α_2) can be integrated. The problem, therefore, can be reduced to the determination of one variable. Other variables can be solved in quadratures. To complete the solution, it remains to fit boundary conditions. This problem, however, is more conveniently addressed by an alternative solution procedure.

B. $\Gamma \in \mathbb{R}$: The second procedure

Another convenient method for solution of 4WM equations is based on the linearization procedure (the replacement of the "time" variable z by the variable $\theta(z) = \int_0^z |\mu(z')|dz' + \theta_0$). Then (2.7) remains the same, but the matrix $\mathbf{m} \to \tilde{\mathbf{m}}$ becomes constant [$\mu \to \nu = \exp(i\phi)$]. The explicit solution of Eqs.(2.7) is now

$$
|\psi_j(\theta)\rangle = \begin{pmatrix} c(\sigma, \Theta) & \sigma\nu s(\sigma, \Theta) \\ -\bar{\nu} s(\sigma, \Theta) & c(\sigma, \Theta) \end{pmatrix} |\psi_j(\theta_0)\rangle, \tag{4.11}
$$

where $\Theta = \theta - \theta_0$ and θ_0 is to be determined from the boundary conditions. The subscript 0 stands for the quantities evaluated at $z = 0$. The matrix in Eq. (4.11) explicitly displays the SU nature of the symmetry of solutions, and allows for an easy identification of Euler angles for the problem: $\alpha = \phi$, $\beta = 2\theta$, $\gamma = -\phi$. In this formulation (real Γ) only one independent variable (θ) is found necessary. The angle ϕ is fixed by the boundary conditions.

The evaluation of θ_0 is facilitated by writing $|Q|$ and q_5 in terms of θ:

$$
|Q| = \sqrt{q}s(\sigma, 2\theta)/2, \quad q_5 = -\sigma\sqrt{q}c(\sigma, 2\theta). \tag{4.12}
$$

The form of the solution is different in the two geometries, since I is constant in TG, whereas it is not in RG. At this point the symmetry in treating the two geometries is broken. The solution of Eq. (4.5) is

$$
\tan(\theta) = \tan(\theta_0)\exp\left(\sqrt{q}\Gamma z/I\right), \tag{4.13}
$$

in TG, and

$$
\sinh(2\theta) = \sinh(2\theta_0)\exp(\Gamma z), \tag{4.14}
$$

in RG. The procedure for evaluation of θ_0 is also different in the two geometries. We first present the TG case.

The angle θ_0 is found when boundary conditions are applied to the solution given by Eq. (4.13). The conditions are that the fields are given at the opposite faces of the crystal: $A_j(z = 0 \text{ or } z = d) = C_j$. In OPC $C_3 = 0$. Using these conditions, a number of auxiliary quantities is defined:

$$
\begin{aligned}
u &= |C_4|^2 - |C_1|^2 + |C_2|^2, \\
v &= |C_4|^2 - |C_1|^2 - |C_2|^2, \\
p &= 2C_1\bar{C}_4\exp(-i\phi), \\
\alpha &= \exp\left(-\sqrt{q}\Gamma d/I\right),
\end{aligned} \tag{4.15}
$$

(all real), and a shorthand notation is introduced:

$$
x = \tan(\theta_d - \theta_0), \quad y = \tan(\theta_d + \theta_0). \tag{4.16}
$$

There exists a rational relation connecting x and y:

$$
y = \frac{ux + p}{v - px}, \quad x = \frac{vy - p}{u + py}. \tag{4.17}
$$

It is seen that x and y depend on θ_0 and [through Eq. (4.13)] on q. However, there also exists a relation expressing x (and likewise y) only through q:

$$x = -\frac{c}{q-a} \pm \left[\left(\frac{c}{q-a} \right)^2 - \frac{q-b}{q-a} \right]^{1/2}, \qquad (4.18)$$

where $a = p^2 + u^2$, $b = p^2 + v^2$, $c = p(u-v) = 2p|C_2|^2$. This relation is used to write an implicit equation for q:

$$\alpha = \xi^2 - \eta^2, \qquad (4.19)$$

where $\xi = (1-\alpha)/2x$, $\eta = (1+\alpha)/2y$. Thus, given the boundary conditions, Eq. (4.19) is to be solved numerically, to determine q. Given q, x and y are found, and θ_0 evaluated from the relation

$$\tan(\theta_0) = \frac{\alpha}{\xi + \eta}. \qquad (4.20)$$

This completes the TG procedure.

For RG, one finds two expressions for the modulus of the grating $|Q|$ at $z = 0$:

$$\begin{aligned} |Q_0| &= \tanh(\Theta_d)|C|^2/(e-1) = \\ &= |p| \mathrm{sech}\left(|C_1|^2 + |C_4|^2\right), \end{aligned} \qquad (4.21)$$

where $|C|^2 = \sum |C_i|^2$, $e = \exp(\Gamma d)$, and now $|p| = |\bar{C}_2 C_4|$. This yields an expression for $\sinh(\Theta_d)$:

$$\sinh(\Theta_d) = \frac{|p|(e-1)}{e\left(|C_1|^2 + |C_4|^2\right) + |C_2|^2}. \qquad (4.22)$$

Using Eqs. (4.21) and (4.22), an expression for $\tanh(2\theta_0)$ is obtained:

$$\tanh(2\theta_0) = \frac{\sinh(2\Theta_d)}{e - \cosh(2\Theta_d)}. \qquad (4.23)$$

This completes the RG procedure.

C. $\Gamma \in i\mathbb{R}$

It is useful to note that beside the equations (2.39) and (2.43), one can derive an equation for the intensity

$$II' = 2(\sigma - 1)\mathrm{Re}\ (\Gamma)|Q|^2. \qquad (4.24)$$

From these equations it follows that in the Γ imaginary case a number of additional quantities is constant: $|Q|$, q_5, I. The equation for the phase can be recast as

$$I \partial_z \arg(Q) = -\tilde{\Gamma} q_5, \qquad (4.25)$$

where $\Gamma \equiv: i\tilde{\Gamma}$, and solved:

$$\arg(Q(z)) = \arg(Q(0)) - \tilde{\Gamma}\frac{q_5}{I} z. \qquad (4.26)$$

Since $|\mu(z)| = |\Gamma Q(z)|/I(z) = |\tilde{\Gamma}||Q|/I$ is constant, one obtains an explicit expression

$$\mu(z) = |\mu_0| \exp(i\phi_0 - i\Omega z), \qquad (4.27)$$

where $|\mu_0| :\equiv |\tilde{\Gamma}||Q|/I$, $\phi_0 :\equiv \pi/2 + \arg\tilde{\Gamma} + \arg(Q(0))$ and $\Omega :\equiv \tilde{\Gamma} q_5/I$.

Now the evolution matrix $\mathbf{m}(z)$ from the spinor EOM (2.7) has the form

$$\mathbf{m}(z) = \begin{pmatrix} 0 & \sigma|\mu_0|e^{i\phi_0 - i\Omega z} \\ -|\mu_0|e^{-i\phi_0 + i\Omega z} & 0 \end{pmatrix}, \qquad (4.28)$$

and the formal solutions of EOM (2.7) are

$$|\psi_i(z)\rangle = \mathbf{U}(z)|\psi_i(0)\rangle. \qquad (4.29)$$

The $\mathbf{U}(z)$ matrix is the *ordered exponential* (see Appendix D for a discussion) of $\mathbf{m}(z)$:

$$\mathbf{U}(z) = \left(\exp\left(\int_0^z dz' \mathbf{m}(z') \right) \right)_+ \qquad (4.30)$$

where the *plus* subscript indicates the path-ordered nature of the exponential. In practice, to obtain the explicit form of $\mathbf{U}(z)$ in terms of non-ordered quantities, one has to solve the initial value problem (IVP)

$$\begin{aligned} \partial_z \mathbf{U}(z) &= \mathbf{m}(z)\mathbf{U}(z), \\ \mathbf{U}(0) &= \mathbf{1}. \end{aligned} \qquad (4.31)$$

For the specific $\mathbf{m}(z)$ the explicit solution to this IVP is found (see Appendix E):

$$\begin{aligned} U_{11} &= \exp\left(-i\frac{\Omega z}{2}\right)\left[\cos\left(\frac{\Xi z}{2}\right) + i\frac{\Omega}{\Xi}\sin\left(\frac{\Xi z}{2}\right)\right], \\ U_{12} &= i\frac{2\sigma Q}{\sqrt{q}}\exp\left(-i\frac{\Omega z}{2}\right)\sin\left(\frac{\Xi z}{2}\right), \\ U_{21} &= -i\frac{2\sigma\bar{Q}}{\sqrt{q}}\exp\left(i\frac{\Omega z}{2}\right)\sin\left(\frac{\Xi z}{2}\right), \\ U_{22} &= \exp\left(i\frac{\Omega z}{2}\right)\left[\cos\left(\frac{\Xi z}{2}\right) - i\frac{\Omega}{\Xi}\sin\left(\frac{\Xi z}{2}\right)\right], \end{aligned} \qquad (4.32)$$

where $\Xi :\equiv \tilde{\Gamma}\sqrt{q}/I$ and $q :\equiv q_5^2 + 4\sigma|Q|^2$. It is easy to check that $\det \mathbf{U}(z) = 1$.

In this manner, for known initial values $|\psi_i(0)\rangle$, the full solution at later "times" $z > 0$ is given by Eq. (4.29). However, by the nature of the 4WM system, one knows only the part of initial conditions. The system represents a split boundary value problem.

• For the TG case the beams $B_1 = A_1$ and $B_3 = A_4$ are entering the crystal sample from the $z = 0$ side, while other two beams $B_2 = A_2$ and $B_4 = A_3$ are coming from the $z = d$ side. Thus, only $|\psi_1\rangle$ is determined at $z = 0$, while $|\psi_2\rangle$ has a fixed value at $z = d$ boundary:

$$|\psi_1(0)\rangle = \begin{pmatrix} C_1 \\ C_3 \end{pmatrix}, \ |\psi_2(d)\rangle = \begin{pmatrix} C_4 \\ -\sigma C_2 \end{pmatrix}. \qquad (4.33)$$

In this way:

$$\begin{aligned} |\psi_1(z)\rangle &= \mathbf{U}(z)|\psi_1(0)\rangle, \\ |\psi_2(z)\rangle &= \mathbf{U}(z)\mathbf{U}(d)^\dagger|\psi_2(d)\rangle. \end{aligned} \qquad (4.34)$$

• For the RG case only $B_1 = A_1$ and $B_4 = A_4$ are known at the $z = 0$ boundary: $B_1(0) = C_1$, $B_4(0) = C_4$, while the remaining two field variables are given at the $z = d$ boundary: $B_2(d) = C_2$, $B_3(d) = C_3$. This means that both spinors $|\psi_i\rangle$ $(i = 1, 2)$ are satisfying the mixed boundary conditions, where one component satisfies the initial value condition (at $z = 0$) and the other component satisfies the final boundary condition (at $z = d$):

$$
\begin{aligned}
|\psi_1(0)\rangle &= \begin{pmatrix} C_1 \\ B_3(0) \end{pmatrix}, \quad |\psi_1(d)\rangle = \begin{pmatrix} B_1(d) \\ C_3 \end{pmatrix}, \\
|\psi_2(0)\rangle &= \begin{pmatrix} C_4 \\ -\sigma B_2(0) \end{pmatrix}, \quad |\psi_2(d)\rangle = \begin{pmatrix} B_4(d) \\ -\sigma C_2 \end{pmatrix},
\end{aligned}
\tag{4.35}
$$

where $B_3(0)$, $B_1(d)$, $B_2(0)$ and $B_4(d)$ are unknown. In order to determine them, one has to use the evolution formula Eq. (4.29) to express the unknown boundary values in terms of the known ones. After some simple algebra one obtains

$$
\begin{aligned}
B_1(d) &= \frac{C_1 + U_{12}(d)C_3}{U_{22}(d)}, \\
B_3(0) &= \frac{C_3 - U_{21}(d)C_1}{U_{22}(d)}, \\
B_4(d) &= \frac{C_4 - \sigma U_{12}(d)C_2}{U_{22}(d)}, \\
B_2(0) &= \frac{C_2 + \sigma U_{21}(d)C_4}{U_{22}(d)}.
\end{aligned}
\tag{4.36}
$$

Here the unimodality condition $\det \mathbf{U}(z) = 1$ $(\forall z)$ was used. This concludes the solution procedure for both geometries.

V. THE (PSEUDO)HAMILTONIAN STRUCTURE OF 4WM

Considering again the form of the spinorial EOM (2.7), one may notice that (in the TG with $\mu \in \mathbb{R}$ case) the matrix $\mathbf{m}$ is antisymmetric, resembling the symplectic matrix used in the Hamiltonian formalism for mechanical systems. This notice gives rise to the question whether it is posible to reformulate the 4WM system as a Hamiltonian system. In this section one possible approach to the problem is considered. First the neccessary general definitions are given, and then the specifics of the 4WM system are discussed.

A. Preliminaries

For the formulation of the Hamiltonian formalism [10] one needs a phase space in the form of a smooth manifold V and a closed nondegenerate differential 2-form $\hat{F}$ (the *field-strength* form) defined on it, which endows a symplectic structure on V. In the phase space with the canonical coordinates $\{q^1, ...q^D, p_1, ...p_D\}$ (q^i not to be confused with the conserved quantities), the canonical form of $\hat{F}$ is

$$
\hat{F} = \sum_{k=1}^{D} dq_k \wedge dp_k. \tag{5.1}
$$

The 2-form $\hat{F}$ sets up an isomorphism between the tangent space TV and the cotangent space $^*TV$. Denote the inverse mapping by $\hat{J} : {}^*TV \to TV$. In the canonical coordinates $\hat{J}$ has the form

$$
\hat{J} = \sum_{k=1}^{D} \partial_{q^k} \wedge \partial_{p_k}. \tag{5.2}
$$

In a general system of coordinates $\{x^\mu | \mu = \overline{1, 2D}\}$, the forms of $\hat{F}$ and $\hat{J}$ become

$$
\begin{aligned}
\hat{F} &= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \\
\hat{J} &= \frac{1}{2} J^{\mu\nu} \partial_\mu \wedge \partial_\nu,
\end{aligned}
\tag{5.3}
$$

with mutually inverse skew-symmetric matrices $\mathbf{F} = (F_{\mu\nu})$ and $\mathbf{J} = (J^{\mu\nu})$. Here the summation over the repeated greek indices is assumed. The matrix $\mathbf{J}$ is known as the *symplectic* matrix.

Physical quantities are smooth functions on V, forming the space $C^\infty(V)$. A Poisson bracket is defined in $C^\infty(V)$, generating a Lie-algebra structure

$$
\{f, g\}_{PB} = J^{\mu\nu} \partial_\mu f \partial_\nu g. \tag{5.4}
$$

In the canonical coordinates this means

$$
\{f, g\}_{PB} = \sum_{k=1}^{D} (\partial_{q^k} f \partial_{p_k} g - \partial_{q^k} g \partial_{p_k} f). \tag{5.5}
$$

The Poisson bracket is bilinear, skew-symmetric, and obeys the Jacobi identity, which is equivalent to the closeness of the 2-form $\hat{F}$: $d\hat{F} = 0$. Later more will be elaborated on this condition.

The dynamics is determined by the choice of the Hamilton function H on V. The external differential dH is a covector field (1-form), and $\hat{J} \cdot dH$ is the corresponding Hamilton's vector field on V. The Hamilton equation of motion is specified by equating the tangent vector field $\dot{\vec{x}} :\equiv \dot{x}^\mu \partial_\mu$ with the Hamilton's vector field:

$$
\dot{\vec{x}} = \hat{J} \cdot dH. \tag{5.6}
$$

The Poisson bracket $\{f, g\}_{PB}$ may now be represented by the action of the covector df on Hamilton's vector field $\hat{J} \cdot dg$: $\{f, g\}_{PB} = df(\hat{J} \cdot dg)$. Therefore, the derivative of function f in the direction of Hamilton's vector field $\hat{J} \cdot dH$ is in fact $\{F, H\}_{PB}$. Hence, the Hamilton equation (5.6) can be written as $\dot{f} = \{f, H\}_{PB}$ for an arbitrary function f. Since the coordinate functions

$\{q^1, ...q^D, p_1, ...p_D\}$ form a complete basis, the equations

$$
\begin{aligned}
\dot{q}^k &= \{q^k, H\}_{PB} = \partial_{p_k} H, \\
\dot{p}_k &= \{p_k, H\}_{PB} = -\partial_{q^k} H,
\end{aligned} \tag{5.7}
$$

form a closed system. These are the canonical Hamilton equations of motion.

B. Four-wave mixing

In an attempt to cast the 4WM system in the Hamiltonian form, one encounters several problems.

First, there is no clear choice of the Hamiltonian (Hamilton's function) $H(x)$. It can be an arbitrary real function of the full set of conserved quantities: $H(x) = h(q(x))$. For example, for the general $\Gamma \in \mathbb{C}$, one can identify three convenient families of Hamiltonians

$$
\begin{aligned}
H_{Q1,\epsilon} &= q_1 + \epsilon q_2, \\
H_{Q3\lambda} &= \lambda q_3 + \bar{\lambda}\bar{q}_3, \\
H_{Q4\lambda} &= \lambda q_4 + \bar{\lambda}\bar{q}_4,
\end{aligned} \tag{5.8}
$$

where $\epsilon = \pm 1$ and $|\lambda| = 1$. Clearly, these families are not exhausting all the possible choices, even among the Hamiltonians that are linear in the regular IOM.

The corresponding symplectic matrices are

$$
\mathbf{J}_{q1\epsilon} = \begin{pmatrix}
 & & & & & & \mu & \\
 & & & & & & & \epsilon\bar{\mu} \\
 & & & & -\bar{\mu} & & & \\
 & & & & & -\epsilon\mu & & \\
 & & \bar{\mu} & & & & & \\
 & & & \epsilon\mu & & & & \\
-\mu & & & & & & & \\
 & -\epsilon\bar{\mu} & & & & & &
\end{pmatrix}, \tag{5.9}
$$

$$
\mathbf{J}_{q3\lambda} = \begin{pmatrix}
 & & & & & -\sigma\bar{\lambda}\mu & & \\
 & & & & \sigma\lambda\bar{\mu} & & & \\
 & & & & & & & -\bar{\lambda}\bar{\mu} \\
 & & & & & & \lambda\mu & \\
 & -\sigma\lambda\bar{\mu} & & & & & & \\
\sigma\bar{\lambda}\mu & & & & & & & \\
 & & & -\lambda\mu & & & & \\
 & & \bar{\lambda}\bar{\mu} & & & & &
\end{pmatrix},
$$

$$
\mathbf{J}_{q4\lambda} = \begin{pmatrix}
 & & & \bar{\lambda}\mu & & & & \\
 & & \bar{\lambda}\bar{\mu} & & & & & \\
 & -\bar{\lambda}\bar{\mu} & & & & & & \\
-\bar{\lambda}\mu & & & & & & & \\
 & & & & & & & \lambda\bar{\mu} \\
 & & & & & & \lambda\mu & \\
 & & & & & -\lambda\mu & & \\
 & & & & -\lambda\bar{\mu} & & &
\end{pmatrix}.
$$

The matrix elements that are not written explicitly, are zero. Note that for these three families (subscript K is the index of each individual family):

$$
\mathbf{J}_K^{-1} = \frac{1}{|\mu|^2}\mathbf{J}_K^\dagger, \qquad \mathbf{J}_K^T = -\mathbf{J}_K. \tag{5.10}
$$

These matrices are chosen in such a way to satisfy the three basic requirements: they are antisymmetric, non-singular (in the matrix sense), and they give the same EOM

$$
J_K^{\mu\nu}\partial_\nu H_K = f^\mu(x),
$$

where $f^\mu(x)$ is the right-hand side of the EOM (1.4): $\dot{x}^\mu = f^\mu$. The fact that all Hamiltonian structures must reproduce the same dynamical equations (1.4) means that for any two structures $(H_A, \mathbf{J}_A)$ and $(H_B, \mathbf{J}_B)$ one can write:

$$
{}^\dagger\nabla_{BA} H_A = \nabla H_B, \tag{5.11}
$$

where ${}^\dagger\nabla_{BA} :\equiv \mathbf{R}_{BA} \cdot \nabla$ defines *the cogradient* and $\mathbf{R}_{BA} :\equiv \mathbf{J}_B^{-1}\mathbf{J}_A$ is *the recursion matrix*, connecting the Hamiltonian structures (A) and (B).

Equation (5.11) defines the mapping from the first Hamiltonian structure (A) to the second one (B). It is interesting to assume for a moment that there exists some function H_C whose gradient is the BA-cogradient of H_B. If such a function exists, it will be an integral of motion:

$$
\begin{aligned}
\partial_z H_C &= \dot{x}^\mu \partial_\mu H_C = \\
&= J_B^{\mu\nu}\partial_\nu H_B (J_B^{-1})_{\mu\alpha} J_A^{\alpha\beta}\partial_\beta H_B = \\
&= -\partial_\alpha H_B J_A^{\alpha\beta}\partial_\beta H_B \equiv 0.
\end{aligned}
$$

Thus, the conserved quantity H_C, if it exists, is a new Hamiltonian of the system, and the corresponding symplectic matrix is $\mathbf{J}_C :\equiv \mathbf{R}_{AB}\mathbf{J}_B = \mathbf{J}_B\mathbf{J}_A^{-1}\mathbf{J}_B$.

One can continue along the same lines, defining a series of conserved quantities ("Hamiltonians") and the corresponding symplectic matrices:

$$
\begin{array}{ccccccc}
H_A & \to & H_B & \to & H_C & \to & \cdots \\
\hat{J}_A & \to & \hat{J}_B & \to & \hat{J}_C & \to & \cdots
\end{array} \tag{5.12}
$$

The sequence terminates when the Hamiltonians start repeating themselves (i.e. they became linear combinations of the previous members of the sequence). This type of sequence of the Hamiltonians and the corresponding symplectic matrices is common in the two-dimensional integrable systems. There exists a *multi-Hamiltonian property* of integrable systems, whereby the chain of Hamiltonians is (usually) non-terminating, leading to an infinite set of non-equivalent IOM, and, thus, to the complete integrability of the system.

In 4WM one expects that all such sequences, if they exist at all, should terminate after a few terms. For example, consider the two structures $(H_A :\equiv H_{Q1,+1} = q_1 + q_2, J_A)$ and $(H_B :\equiv H_{Q1,-1} = q_1 - q_2, J_B)$, where J_A and J_B are the special cases of (5.9). The recursive matrix is $\mathbf{R}_{BA} :\equiv \mathbf{J}_B^{-1}\mathbf{J}_A = \mathrm{diag}(1, -1, 1, -1, 1, -1, 1, -1)$

and the basic identity is ${}^{\dagger}\nabla_{BA} H_A = \nabla H_B$. The definition of the induced H_C is

$$\nabla H_C :\equiv {}^{\dagger}\nabla_{BA} H_B, \tag{5.13}$$

and its solution is $H_C = q_1 + q_2$. Thus $H_C = H_A$ and the sequence is periodic: $H_A \to H_B \to H_A \to \cdots$.

Each Hamiltonian structure $(H_K, \mathbf{J}_K)$ has the corresponding Poisson bracket:

$$\{f, g\}_K :\equiv (\nabla f)^T \mathbf{J}_K (\nabla g). \tag{5.14}$$

This bracket is antisymmetric $\{g, f\}_K = -\{f, g\}_K$, and can be characterized by the set of basic brackets:

$$\{x^\mu, x^\nu\}_K = J_K^{\mu\nu}. \tag{5.15}$$

However, a bracket so defined does not satisfy the expected Jacobi identity $\{f, \{g, h\}_K\}_K + \text{cyclic} \equiv 0$. The tensor of *non-Jacobianity* measures the violation of the Jacobi identity:

$$\begin{aligned} \Omega_{(K)}^{\mu\nu\alpha} &:\equiv \{\{x^\mu, x^\nu\}_K, x^\alpha\}_K + cyclic = \\ &= (\partial_\sigma J_K^{\mu\nu}) J^{\sigma\alpha} + cyclic. \end{aligned} \tag{5.16}$$

This tensor is essentially the same as the tensor of deflection from the Bianchi identity, $\omega :\equiv d\hat{F}$ (i.e. $\omega_{\mu\nu\alpha} :\equiv \partial_\mu F_{\nu\alpha} + cyclic$):

$$\Omega_{(K)}^{\mu\nu\alpha} = J_K^{\mu\mu'} J_K^{\nu\nu'} J_K^{\alpha\alpha'} \omega_{(K)\mu'\nu'\alpha'}. \tag{5.17}$$

In the case of non-singular $\mathbf{J}$, the Jacobi condition is equivalent to the Bianchi identity.

In the system at hand, the fact that $\Omega_{(K)}$ is not disappearing is the consequence of the non-constancy of μ. For example, take again the Hamiltonian $H = H_{Q1,+1} = q_1 + q_2$. Then

$$\mathbf{J} = \mu \mathbf{E}_+ + \bar{\mu} \mathbf{E}_-, \tag{5.18}$$

where

$$\mathbf{E}_+ = \begin{pmatrix} & & & & & & 1 & \\ & & & & & & & 0 \\ & & & & 0 & & & \\ & & & & & -1 & & \\ & & 0 & & & & & \\ & & & 1 & & & & \\ -1 & & & & & & & \\ & 0 & & & & & & \end{pmatrix},$$

$$\mathbf{E}_- = \begin{pmatrix} & & & & & & 0 & \\ & & & & & & & 1 \\ & & & & -1 & & & \\ & & & & & 0 & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ 0 & & & & & & & \\ & -1 & & & & & & \end{pmatrix}.$$

Note that $\mathbf{E}_\pm^T = -\mathbf{E}_\pm$, $\mathbf{E}_\pm \cdot \mathbf{E}_\mp = 0$, and

$$(\mathbf{E}_\pm)^2 = -\frac{1}{2} [\mathbf{1}_8 \pm \sigma_3 \otimes \sigma_3 \otimes \sigma_3]. \tag{5.19}$$

As a simple consequence of these expressions, the following identity is valid

$$(\alpha \mathbf{E}_+ + \beta \mathbf{E}_-)^{-1} = -\frac{1}{\alpha} \mathbf{E}_+ - \frac{1}{\beta} \mathbf{E}_-, \tag{5.20}$$

for arbitrary (nonzero) α and β. This identity is used to evaluate the "field strength" matrix:

$$\mathbf{F} = -\frac{1}{\mu} \mathbf{E}_+ - \frac{1}{\bar{\mu}} \mathbf{E}_-. \tag{5.21}$$

From this, one obtains

$$\begin{aligned} \omega_{\mu\nu\alpha} &= \frac{1}{\mu^2} [(\partial_\alpha \mu)(E_+)_{\mu\nu} + cyclic] + \\ &\quad + \{\mu \to \bar{\mu}, E_+ \to E_-\}. \end{aligned} \tag{5.22}$$

For example, $\omega_{B_1, B_4, \bar{B}_2} = -\frac{1}{\mu^2} \partial_1 \mu$. Thus, the Bianchi identity is clearly broken, and one can not find the potential $\mathcal{A}_\mu(x)$ such that $F_{\mu\nu}(x)$ is its strength tensor.

One can see the dual nature of the same obstacle, expressed in terms of the Poisson bracket, in the following way. The basic brackets are

$$\begin{aligned} \{B_1, \bar{B}_3\}_{PB} &= \mu, \\ \{B_2, \bar{B}_4\}_{PB} &= \bar{\mu}. \end{aligned} \tag{5.23}$$

and one non-vanishing component of Ω is

$$\Omega^{B_1 \bar{B}_3 B_2} = -\bar{\mu} \bar{\partial}_4 \mu. \tag{5.24}$$

The full list of non-vanishing components of Ω is given in the Appendix F. Thus, the Poisson bracket is not self-consistent: one can not apply it consecutively on the phase space without running into inconsistencies. This is the second, and much more serious problem with the presented approach to casting the 4WM system in the Hamiltonian form. One can say that the 4WM system has a *pseudo-Hamiltonian structure*.

If all components of ω were zero, one would be able to find the potential functions $\mathcal{A}_\mu$ of strength tensor $F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$. Then the system could be formulated as the Lagrangian one, with the action functional $S(z_1, z_2) :\equiv \int_{z_1}^{z_2} dz L(x, \dot{x})$, where the Lagrangian function is

$$L(x, \dot{x}) :\equiv \dot{x}^\mu \mathcal{A}_\mu(x) - H(x). \tag{5.25}$$

The Euler-Lagrange EOM corresponding to this Lagrangian are the Hamilton equations (5.6). The elements of the Lagrangian formalism are provided in Appendix G.

Since $\omega \neq 0$, one may search for the solutions in the form

$$F_{\mu\nu} = f(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu), \tag{5.26}$$

which leads to

$$(\partial_\alpha f)F_{\mu\nu} + cyclic = f\omega_{\alpha\beta\gamma}. \tag{5.27}$$

The direct consequence of this equation is

$$\partial_\alpha \ln f = \frac{1}{2D-2}\omega_{\alpha\mu\nu}J^{\nu\mu} \tag{5.28}$$

where $2D = 8$ in the 4WM system. After some algebra one derives

$$\partial_\alpha \ln f = -\partial_\alpha \ln|\mu| + \frac{i}{3}\left(\sigma_3 \otimes \sigma_3 \otimes \sigma_3\right)_\alpha{}^\nu \partial_\nu \phi. \tag{5.29}$$

where $\phi = \arg(\mu)$.

C. The $\Gamma \in \mathbb{R}$ case

If ϕ is constant (i.e. $\Gamma \in \mathbb{R}$), the solution of Eq. (5.29) is $f = 1/|\mu|$. Then

$$\tilde{F}_{\mu\nu} = -\bar{\nu}\left(E_+\right)_{\mu\nu} - \nu\left(E_-\right)_{\mu\nu},$$

where $\tilde{F}_{\mu\nu} :\equiv F_{\mu\nu}/f = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$, and the solution for the potential $\mathcal{A}_\mu$ is

$$\mathcal{A}_\mu = -\frac{1}{2}\left[\bar{\nu}\left(E_+\right)_{\mu\nu} + \nu\left(E_-\right)_{\mu\nu}\right]x^\nu, \tag{5.30}$$

with $\nu = \exp(i\phi)$ (not to be confused with the index ν).

To construct the action for this case one has to go one step back. The factorization of $f = 1/|\mu(x)|$ from $F_{\mu\nu}$ is equivalent to the introduction of a new time parameter $\theta(z) :\equiv \int_0^z dz'\mathcal{M}(z') + const.$ into EOM, where $\mathcal{M}(z) :\equiv |\mu\left(x(z)\right)|$ is the on-shell value of $|\mu(x)|$. Thus

$$\tilde{F}_{\mu\nu}\frac{dx^\nu}{d\theta} = \partial_\mu H(x).$$

The constant "field-strength" form $\hat{\tilde{F}}$ is closed, and its tensor of non-Jacobianity $\tilde{\omega}$ disappears. So, one can construct the action in the rescaled time

$$S(\theta_1,\theta_2) = \int_{\theta_1}^{\theta_2} d\theta\left[\frac{1}{2}x^\mu \tilde{F}_{\mu\nu}\frac{dx^\nu}{d\theta} - H(x)\right]. \tag{5.31}$$

Further transformation of this action

$$\begin{aligned} S &= \int_{z_1}^{z_2} dz\mathcal{M}(z)\left[\frac{1}{2}x^\mu\tilde{F}_{\mu\nu}\frac{1}{\mathcal{M}(z)}\frac{dx^\nu}{dz} - H(x)\right] = \\ &= \int_{z_1}^{z_2} dz\left[\frac{1}{2}x^\mu\tilde{F}_{\mu\nu}\frac{dx^\nu}{dz} - \mathcal{M}(z)H(x)\right], \end{aligned}$$

leads to the Lagrangian

$$\begin{aligned} L &= -\frac{1}{2}\dot{x}^\mu\left[\bar{\nu}\left(E_+\right)_{\mu\nu} + \nu\left(E_-\right)_{\mu\nu}\right]x^\nu + \\ &\quad -\mathcal{M}(z)(q_1+q_2), \end{aligned} \tag{5.32}$$

Note that if in the above expression $\mathcal{M}(z)$ is directly replaced by $|\mu(x)|$, the obtained corresponding variation equations are wrong.

In the $\Gamma \in \mathbb{R}$ case the set of IOM is enlarged by the "exceptional" IOM $\{w_{1-6}\}$ (and their complex conjugates), and one can construct some additional families of (linear in IOM) Hamiltonians ($\epsilon = \pm 1$, $|\lambda| = 1$, $|\theta| = 1$):

$$\begin{aligned} H_{W1,\epsilon} &= w_1 + \epsilon\sigma w_2, \\ H_{W3,\lambda} &= \lambda w_3 + \bar{\lambda}\bar{w}_3, \\ H_{W5,\lambda} &= \lambda w_5 + \bar{\lambda}\bar{w}_5, \\ H_{W4,\lambda\theta} &= \frac{1}{2}(\lambda w_4 + \bar{\lambda}\bar{w}_4 + \theta w_6 + \bar{\theta}\bar{w}_6). \end{aligned} \tag{5.33}$$

The corresponding symplectic matrices are

$$\mathbf{J}_{W1\epsilon} = \begin{pmatrix} & & & & \sigma & & & \\ & & & & & \epsilon\sigma & & \\ & & & & & & 1 & \\ & & & & & & & \epsilon \\ -\sigma & & & & & & & \\ & -\epsilon\sigma & & & & & & \\ & & -1 & & & & & \\ & & & -\epsilon & & & & \end{pmatrix},$$

$$\mathbf{J}_{W3\lambda} = \begin{pmatrix} & & & & & & & -\lambda\sigma \\ & & & & & & -\bar{\lambda}\sigma & \\ & & & & & \lambda\sigma & & \\ & & & & \bar{\lambda}\sigma & & & \\ & & & -\bar{\lambda}\sigma & & & & \\ & & -\lambda\sigma & & & & & \\ & \bar{\lambda}\sigma & & & & & & \\ \lambda\sigma & & & & & & & \end{pmatrix},$$

$$\mathbf{J}_{W5\lambda} = \begin{pmatrix} & -\sigma\bar{\lambda} & & & & & & \\ \sigma\bar{\lambda} & & & & & & & \\ & & & -\bar{\lambda} & & & & \\ & & \bar{\lambda} & & & & & \\ & & & & & -\sigma\lambda & & \\ & & & & \sigma\lambda & & & \\ & & & & & & & \lambda \\ & & & & & & -\lambda & \end{pmatrix},$$

$$\mathbf{J}_{W4\lambda\theta} = \begin{pmatrix} & & \bar{\lambda} & & & & & \\ & & & \sigma\bar{\theta} & & & & \\ -\bar{\lambda} & & & & & & & \\ & -\sigma\bar{\theta} & & & & & & \\ & & & & & & \lambda & \\ & & & & & & & \sigma\theta \\ & & & & -\lambda & & & \\ & & & & & -\sigma\theta & & \end{pmatrix}.$$

These matrices $\mathbf{J}_{W\cdots}$ do not depend on μ, i.e. they are constant. So, their Poisson brackets satisfy the Jacobi identity. This is the case not only for the real, but also for the complex coupling Γ. However, the Hamiltonians (5.33) are not IOM for the complex Γ.

To sum up the results, for a general Γ two types of pseudo-Hamitonian structures exist:

• $\underline{(H_Q, \mathbf{J}_Q)}$: Hamiltonians H_Q are linear in regular IOM. They are conserved quantities in general case, but the corresponding field-strength forms $\hat{F}_Q$ are non-closed

$d\hat{F}_Q \neq 0$. The defect of this type is the non-closeness of its symplectic structure.

• $\underline{(H_W, \mathbf{J}_W)}$: Hamiltonians H_W are linear in exceptional ($\Gamma \in \mathbb{R}$) IOM. The corresponding field-strength forms $\hat{F}_W$ are constant and closed in general case. The defect of this type of structure is the nonconservation of W-Hamiltonians (in general, $\Gamma \in \mathbb{C}$ case).

In the case of real coupling, both defects disappear, the first one after rescaling $z \to \theta(z)$, and the second one because W-Hamiltonians become constant. Then one can construct a consistent Hamiltonian structure for the 4WM system.

VI. CONCLUSIONS

In summary, the algebraic structures of the 4WM equations in PR crystals were studied.

First, the form of the equations of motion was used to group the basic fields into two doublets, leading to the new form of EOM, resembling the Dirac equation in one dimension ("time"), with the field-dependent mass matrix. This lead to the simple procedure for finding the complete set of "regular" (i.e. present in the complex Γ case) integrals of motion. Then an alternative but closely related procedure, based on the Lax pair approach, was used to check the completeness of the obtained set of IOM. Both procedures were extended to the special case $\Gamma \in \mathbb{R}$, to obtain an additional ("exceptional") set of IOM.

Afterwards, the concept of symmetries of the "regular" IOM was defined (the I-symmetries), and the Lie algebras (and groups) corresponding to the linear I-symmetries were found. These are the $su(2)$ symmetry for the transmission gratings and the $su(1,1)$ symmetry for the reflection gratings. The initial doublets of basic fields, which were introduced as a convenience for more compact calculations, turned out to be the fundamental (i.e. spinor) representation of those symmetry algebras. Also, the Lax matrices, constructed from these basic spinors, transform in the regular way, i.e. they form the adjoint representation of the I-symmetries.

In the special case $\Gamma \in \mathbb{R}$ the number of IOM increases, so only the subset of "regular" I-symmetries survives. This is to be expected, since the I-symmetries now have to satisfy a larger set of constraints than in the general ($\Gamma \in \mathbb{C}$) case.

In the second part of the paper another type of symmetries was considered, the symmetries of EOM (the E-symmetries). The corresponding symmetry algebras are the products of several abelian factors (one noncompact $\sim \mathbb{R}^1$, and two compact $\sim u(1)$) and of one $su(1,1)$ factor (for both geometries). The action of these symmetries on the regular IOM was studied and a special kind of Nöther theorem is found to be valid here.

In the special case $\Gamma \in \mathbb{R}$ the number of independent EOM gets smaller, leading to the increase in the number of E-symmetries. This is clearly the opposite behavior to the case of I-symmetries. Further study is necessary to clarify the relation between the "regular" and "exceptional" cases. At the end of this part, the action of the E-symmetries on the I-symmetries was considered (in the "regular" case). The non-abelian factor of E-symmetries commutes with the I-symmetries, and the two $u(1)$ factors act as rotations in the $1-2$ plane of I-symmetries.

As a short excursion from the algebraic orientation of the paper, Section IV is devoted to the solutions of EOM in two "exceptional" cases: $\Gamma \in \mathbb{R}$ and $\Gamma \in i\mathbb{R}$. In both cases it is relatively straightforward to obtain the general solutions (two methods for $\Gamma \in \mathbb{R}$ were presented and one for $\Gamma \in i\mathbb{R}$), but satisfying the boundary conditions characteristic of 4WM geometries required more attention.

In the last Section one possible approach to the Hamiltonian formulation of the 4WM system was discussed. The problems that occurred in that program were twofold: the non-uniqueness of the choice of the Hamiltonian (Hamilton's function), and the non-closeness of the field-strength 2-form. The first problem leads to the recognition of the multi-Hamiltonian nature of the 4WM system, and is not really a problem. It is just the type of the "gauge-symmetry" of EOM. The second problem, however, is the real obstacle to the fulfillment of the program. The structure of this obstacle is topological (the violation of the Jacobi and Bianchi identities). This was studied for one specific "gauge" (the choice of the Hamiltonian), and a special circumstance when this obstacle can be removed was found, essentially corresponding to the $\Gamma \in \mathbb{R}$ case.

The same $\Gamma \in \mathbb{R}$ case was then treated in a different way, leading to the discovery of even bigger space of possible Hamiltonians. The topological obstacle is absent in this case, and nothing prevents a full consistent application of the Hamilton formalism, and identification of the corresponding Hamiltonian action of the system.

Future work: The presented work contains several topics that deserve future attention.

Questions pertaining to the general class of dynamic systems: Clarifying the freedom of choice of Hamiltonian function among IOM; Studying properties of E-symmetries upon the local (i.e. x-dependent) scalings of the dynamic vector field $\vec{F}$; Finding classes of equivalency of the symplectic form $\hat{J}$ (under such scalings) that have the same structure of the tensor of non-Jacobianity ω; etc.

Questions related to the 4WM system in particular: Full relation between the $\Gamma \in \mathbb{R}$ and $\Gamma \in \mathbb{C}$ E-symmetries; Explicit resolution of boundary conditions in $\Gamma \in \mathbb{C}$ case; Extending the theory to *multiple* gratings; etc.

The last question is particularly intriguing. Even in

the case of single gratings, 4WM EOM possess rich algebraic structure. However, the writing of gratings in a photorefractive crystal is a dynamic holographic process, and more than one grating can coexist simultaneously in the same region of the crystal. EOM then contain terms coming from different types of gratings, and the analysis should be much more involved.

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APPENDIX A: THE σ-METRIC ELEMENTARY FUNCTIONS.

The elementary definitions and relations of the c and s functions are listed in this Appendix:

$$\begin{aligned} c(\sigma,x) &:\equiv \cos(\sqrt{\sigma}x) = \begin{cases} \cos(g\alpha), \\ \cosh(\alpha), \end{cases} \\ s(\sigma,x) &:\equiv \sin(\sqrt{\sigma}x)/\sqrt{\sigma} = \begin{cases} \sin(\alpha), \\ \sinh(\alpha), \end{cases} \end{aligned} \tag{A1}$$

where the upper/lower option corespond to $\sigma = +1/-1$ signs.

$$c(\sigma,x)^2 + \sigma s(\sigma,x)^2 = 1, \tag{A2}$$

$$\begin{aligned} 2c(\sigma,x)s(\sigma,x) &= s(\sigma,2x), \\ 2c(\sigma,x)^2 &= 1 + c(\sigma,2x), \\ 2\sigma s(\sigma,x)^2 &= 1 - c(\sigma,2x), \\ c(\sigma,x)^2 - \sigma s(\sigma,x)^2 &= c(\sigma,2x), \end{aligned} \tag{A3}$$

$$\begin{aligned} t(\sigma,x) &:\equiv s(\sigma,x)/c(\sigma,x), \\ ct(\sigma,x) &:\equiv 1/t(\sigma,x), \end{aligned} \tag{A4}$$

$$\begin{aligned} t(\sigma,x) + \sigma ct(\sigma,x) &= 2\sigma/s(\sigma,2x), \\ t(\sigma,x) - \sigma ct(\sigma,x) &= -2\sigma ct(\sigma,2x), \end{aligned} \tag{A5}$$

$$\begin{aligned} c'(\sigma,x) &= -\sigma s(\sigma,x), \\ s'(\sigma,x) &= c(\sigma,x). \end{aligned} \tag{A6}$$

APPENDIX B: AN ALTERNATIVE LAXIAN APPROACH

In order to achieve a sufficient degree of generality, one should use the "big spinor"

$$|\Psi\rangle :\equiv \begin{pmatrix} B_1 \\ \alpha B_3 \\ \beta B_4 \\ \gamma B_2 \end{pmatrix}, \tag{B1}$$

with arbitrary complex numbers α, β and γ. Its evolution equation is

$$\partial_z |\Psi\rangle = \mathcal{N} |\Psi\rangle, \tag{B2}$$

where

$$\mathcal{N} = \begin{pmatrix} 0 & \sigma\mu/\alpha & 0 & 0 \\ -\alpha\bar{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta\mu/\gamma \\ 0 & 0 & \sigma\gamma\bar{\mu}/\beta & 0 \end{pmatrix}. \tag{B3}$$

The problem is to determine the evolving member of the Lax pair.

1. $\mathcal{L} \sim |\Psi\rangle\langle\Psi|$

The matrix $\mathcal{L}$ is searched first in the form $\mathcal{L} = \mathcal{A}|\Psi\rangle\langle\Psi|\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are some constant matrices. Then

$$\partial_z \mathcal{L} = \mathcal{A}\mathcal{N}\mathcal{A}^{-1}\mathcal{L} + \mathcal{L}\mathcal{B}^{-1}\mathcal{N}^\dagger\mathcal{B}. \tag{B4}$$

Require

$$\mathcal{B}^{-1}\mathcal{N}^\dagger\mathcal{B} = -\mathcal{A}\mathcal{N}\mathcal{A}^{-1}, \tag{B5}$$

i.e.

$$\mathcal{N}^\dagger\mathcal{C} = -\mathcal{C}\mathcal{N}, \tag{B6}$$

where $\mathcal{C} :\equiv \mathcal{B}\mathcal{A}$.

The solution of this equations is

$$\mathcal{C} = \begin{pmatrix} \bar{\alpha}\xi_2 & \bar{\alpha}\mu\xi_1 & \bar{\alpha}\xi_4 & \bar{\alpha}\mu\xi_3 \\ -\alpha\bar{\mu}\xi_1 & \frac{\sigma}{\alpha}\xi_2 & \frac{\sigma\gamma\bar{\mu}}{\beta}\xi_3 & -\frac{\beta}{\gamma}\xi_4 \\ \frac{\sigma\bar{\gamma}}{\beta}\xi_6 & \frac{\sigma\bar{\gamma}\mu}{\beta}\xi_5 & \frac{\sigma\bar{\gamma}}{\beta}\xi_8 & \frac{\sigma\bar{\gamma}\mu}{\beta}\xi_7 \\ \alpha\bar{\mu}\xi_5 & -\frac{\sigma}{\alpha}\xi_6 & -\frac{\sigma\gamma\bar{\mu}}{\beta}\xi_7 & -\frac{\beta}{\gamma}\xi_8 \end{pmatrix}. \tag{B7}$$

In the general case of complex Γ the factor μ is non-constant (with respect to z), and some of the parameters ξ have to be set to zero: $\xi_1 = \xi_3 = \xi_5 = \xi_7 = 0$. However, if Γ is real, the phase factor $\nu :\equiv \exp(i\arg\mu)$ is constant, and one may redefine the time variable $z \to \theta$, to absorb the non-constant absolute value $|\mu(z)|$. In effect this permits the full set of non-zero ξ in the above matrix $\mathcal{C}$ (with the replacement $\mu \to \nu$).

Let us consider the general case. The presence of four non-zero ξ parameters indicates the existence of four IOM:

$$\begin{aligned} IOM_{\xi_2} &= \bar{\alpha}(I_1 + \sigma I_3) = \bar{\alpha}q_1, \\ IOM_{\xi_8} &= \beta\bar{\gamma}(I_2 + \sigma I_4) = \beta\bar{\gamma}q_2, \\ IOM_{\xi_4} &= \bar{\alpha}\beta(\bar{B}_1B_4 - \bar{B}_3B_2) = \bar{\alpha}\beta\bar{q}_3, \\ IOM_{\xi_6} &= \sigma\bar{\gamma}(\bar{B}_4B_1 - \bar{B}_2B_3) = \sigma\bar{\gamma}q_3, \end{aligned} \tag{B8}$$

where

$$IOM_{\xi_i} :\equiv \mathrm{Tr}\frac{\partial\mathcal{L}}{\partial\xi_i} = \left\langle \Psi \left| \frac{\partial\mathcal{C}}{\partial\xi_i} \right| \Psi \right\rangle. \tag{B9}$$

These integrals are the already known "regular" IOM. Only q_4 is not obtained in this way. It will be obtained in the next subsection, with a different choice of $\mathcal{L}$.

In the $\Gamma \in \mathbb{R}$ case, there are eight free parameters (ξ) and there should be eight IOM. The first four of them are the "regular" ones $\{q_1, q_2, q_3, \bar{q}_3\}$, and the additional four are:

$$\begin{aligned} IOM_{\xi_1} &= |\alpha|^2(\nu\bar{B}_1B_3 - \bar{\nu}\bar{B}_1B_3) = |\alpha|^2 w_1, \\ IOM_{\xi_7} &= \sigma|\gamma|^2(\nu\bar{B}_4B_2 - \bar{\nu}B_4\bar{B}_2) = -|\gamma|^2 w_2, \\ IOM_{\xi_3} &= \bar{\alpha}\gamma(\nu\bar{B}_1B_2 + \sigma\bar{\nu}\bar{B}_3B_4) = -\sigma\bar{\alpha}\gamma w_3, \\ IOM_{\xi_5} &= -\sigma\alpha\bar{\gamma}\bar{w}_3. \end{aligned} \tag{B10}$$

Here, again, all obtained integrals are already known. They are the elements of the w-set derived in the spinorial approach. The remaining elements of that set will be obtained in the next subsection.

2. $\mathcal{L} \sim |\Psi\rangle \langle\bar{\Psi}|$

Now search for $\mathcal{L}$ in the form $\mathcal{L} = \mathcal{A}\left|\bar{\Psi}\right\rangle \langle\Psi|\,\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are some constant matrices (different from the ones in the previous subsection). Then

$$\partial_z\mathcal{L} = \mathcal{A}\mathcal{N}\mathcal{A}^{-1}\mathcal{L} + \mathcal{L}\mathcal{B}^{-1}\mathcal{N}^T\mathcal{B}, \tag{B11}$$

and the requirement that $\mathcal{L}$ satisfies the Lax-type evolution equation has the form:

$$\mathcal{B}^{-1}\mathcal{N}^T\mathcal{B} = -\mathcal{A}\mathcal{N}\mathcal{A}^{-1}, \tag{B12}$$

i.e.

$$\mathcal{N}^T\mathcal{C} = -\mathcal{C}\mathcal{N}, \tag{B13}$$

where, again, $\mathcal{C} :\equiv \mathcal{B}\mathcal{A}$. The solution to this condition is

$$\mathcal{C} = \begin{pmatrix} \alpha\bar{\mu}\xi_2 & \xi_1 & \alpha\bar{\mu}\xi_8 & \alpha\xi_5 \\ -\xi_1 & \frac{\sigma\mu}{\alpha}\xi_2 & \frac{\sigma\gamma}{\beta}\xi_5 & -\frac{\beta\mu}{\gamma}\xi_8 \\ \frac{\sigma\bar{\mu}}{\beta}\xi_7 & \frac{\sigma\gamma}{\beta}\xi_6 & \frac{\sigma\gamma\bar{\mu}}{\beta}\xi_4 & \xi_3 \\ \alpha\xi_6 & -\frac{\mu}{\alpha}\xi_7 & -\xi_3 & -\frac{\beta\mu}{\gamma}\xi_4 \end{pmatrix}. \tag{B14}$$

In the $\Gamma \in \mathbb{C}$ case one has to set $\xi_2 = \xi_8 = \xi_7 = \xi_4 = 0$ and the remaining ξ parameters give rise to the following four "regular" IOM:

$$\begin{aligned} IOM_{\xi_1} &\equiv 0, \\ IOM_{\xi_3} &\equiv 0, \\ IOM_{\xi_5} &= \alpha\gamma(B_1B_2 + \sigma B_3B_4) = \alpha\gamma q_4, \\ IOM_{\xi_6} &= \alpha\gamma q_4. \end{aligned} \tag{B15}$$

The $\Gamma \in \mathbb{R}$ case has additional w integrals:

$$\begin{aligned} IOM_{\xi_2} &= \alpha(\bar{\nu}B_1^2 + \sigma\nu B_3^2) = \alpha w_4, \\ IOM_{\xi_4} &= \beta\gamma(\sigma\bar{\nu}B_4^2 + \nu B_2^2) = \beta\gamma w_6, \\ IOM_{\xi_8} &= \alpha\beta(\bar{\nu}B_1B_4 - \nu B_3B_2) = \alpha\beta w_5, \\ IOM_{\xi_7} &= \gamma w_5. \end{aligned} \tag{B16}$$

In this way, the full sets of "regular" (q-set) and "exceptional" (w-set) IOMs are reconstructed. The presence of the arbitrary complex constants α, β and γ in the procedure indicates that there are no additional IOM of the bilinear type.

APPENDIX C: HYPERBOLOIDS IN $\mathbb{R}^4$

In $\mathbb{R}^4$ there exist four different types of the normalized "hyperboloids", defined by $y_1^2 + \epsilon_2 y_2^2 + \epsilon_3 y_3^2 + \epsilon_4 y_4^2 = 1$:

	ϵ_2	ϵ_3	ϵ_4
$\mathbb{H}^3_{(4,0)}$	$+1$	$+1$	$+1$
$\mathbb{H}^3_{(3,1)}$	$+1$	$+1$	-1
$\mathbb{H}^3_{(2,2)}$	$+1$	-1	-1
$\mathbb{H}^3_{(1,3)}$	-1	-1	-1

$\mathbb{H}^3_{(4,0)}$ is just another name for the sphere $\mathbb{S}^3$, and $\mathbb{H}^3_{(2,2)}$ is the hyperboloid $\mathbb{H}^3$ relevant for this work.

APPENDIX D: ORDERED EXPONENTIAL

In this appendix some general properties of the matrix $\mathbf{U}(z)$ are discussed.

The basic 4WM EOM (2.7) has a formal solution

$$|\psi_i(z)\rangle = \mathbf{U}(z)\,|\psi_i(0)\rangle\,, \tag{D1}$$

where $\mathbf{U}(z)$ satisfies the initial value problem

$$\begin{aligned} \partial_z\mathbf{U}(z) &= \mathbf{m}(z)\mathbf{U}(z), \\ \mathbf{U}(0) &= \mathbf{1}. \end{aligned} \tag{D2}$$

One can write the formal solution to this equation in the form

$$\mathbf{U}(z) :\equiv \left(\exp\left(\int_0^z dz'\mathbf{m}(z')\right)\right)_+ \tag{D3}$$

which is called *the (Path) Ordered Exponential* (OE). The notion of ordering is referring here to **the right-to-left** multiplication of the factors in the definition of OE:

$$\left(e^{\left(\int_0^z dz'\mathbf{m}(z')\right)}\right)_+ :\equiv \lim_{N\to\infty}\prod_{\alpha=N}^{0} e^{\left(\frac{z}{N}\mathbf{m}\left(\alpha\frac{z}{N}\right)\right)}, \tag{D4}$$

i.e. one alternates the infinitesimal integrations (along the path between the $z' = 0$ and $z' = z$) and exponentiations of, in such a way obtained, infinitesimal matrices.

OE is an entirely different object from the ordinary matrix exponential $\exp(\int_0^z dz'\mathbf{m}(z'))$, where the whole integration along the path is performed first, and then only one exponentiation executed on this integrated matrix. The source of the difference is in the non-commutativity of the matrices $\mathbf{m}(z)$ evaluated at different points.

The methods to evaluate OE are frequently non-exact: one may easily prove that the knowledge of $\mathbf{U}(z)$ for arbitrary $\mathbf{m}(z)$ is equivalent to the knowledge of the solution to the Schrödinger equation for an arbitrary complex potential (and this is known to be a non-solvable problem). However, in the cases when $\mathbf{m}(z)$ has one of the several special forms, the exact solution for $\mathbf{U}(z)$ can be found. Two such cases are encountered in this work:

• for $\Gamma \in \mathbb{R}$ the matrix $\mathbf{m}(z)$ is proportional to the constant matrix $\tilde{\mathbf{m}}(z) = \begin{pmatrix} 0 & \sigma\nu \\ -\bar{\nu} & 0 \end{pmatrix}$, and all commutators $[\mathbf{m}(z), \mathbf{m}(z')]$ are equal to zero. Thus, OE reduces to the ordinary exponential, and the result is displayed in Eq. (4.11).

• for $\Gamma \in i\mathbb{R}$ the matrix $\mathbf{m}(z)$ has the raising and the lowering components that oscillate with the opposite frequencies Ω. The Appendix E gives one possible way to obtain $\mathbf{U}(z)$ for such an $\mathbf{m}(z)$.

In the general case, the OE $\mathbf{U}(z)$ satisfies several simple identities, induced by the properties of $\mathbf{m}(z)$:

• From the tracelessness of $\mathbf{m}(z)$ follows the unimodality condition $\det \mathbf{U}(z) = 1$ $(\forall z)$:

$$\begin{aligned} \det \mathbf{U} &= \lim_{N\to\infty} \prod_{\alpha=N}^{0} \det \left[\exp\left(\frac{z}{N}\mathbf{m}\left(\alpha\frac{z}{N}\right)\right)\right] = \\ &= \lim_{N\to\infty} \prod_{\alpha=N}^{0} \exp\left(\frac{z}{N}\mathrm{Tr}\mathbf{m}\left(\alpha\frac{z}{N}\right)\right) = \\ &= \lim_{N\to\infty} \prod_{\alpha=N}^{0} \exp(0) = 1. \end{aligned} \tag{D5}$$

• From the membership of $\mathbf{m}(z)$ in the Lie algebra $g :\equiv su\left((3+\sigma)/2, (1-\sigma)/2\right)$ it follows that $\mathbf{U}(z)$ is an element of the corresponding Lie group $\mathcal{G} :\equiv SU\left((3+\sigma)/2, (1-\sigma)/2\right)$:

$$\mathbf{U}(z)^{\dagger}\eta\mathbf{U}(z) = \eta, \tag{D6}$$

where the $\eta = \mathbf{n}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}$ is the metric matrix of the algebra g. To check this identity, one should follow the chain of arguments:

$$\begin{aligned} \mathbf{m}\left(\alpha\frac{z}{N}\right) \in g \;&\Rightarrow\; \frac{z}{N}\mathbf{m}\left(\alpha\frac{z}{N}\right) \in g \\ &\Rightarrow\; \exp\left(\frac{z}{N}\mathbf{m}\left(\alpha\frac{z}{N}\right)\right) \in \mathcal{G} \\ &\Rightarrow\; \prod_{\alpha}\exp\left(\frac{z}{N}\mathbf{m}\left(\alpha\frac{z}{N}\right)\right) \in \mathcal{G} \\ &\Rightarrow\; \mathbf{U}(z) \in \mathcal{G}. \end{aligned} \tag{D7}$$

APPENDIX E: ORDERED EXPONENTIAL SOLUTION FOR $\Gamma \in I\mathbb{R}$

In this appendix the initial value problem

$$\begin{aligned} \partial_z \mathbf{U}(z) &= \mathbf{m}(z)\mathbf{U}(z), \\ \mathbf{U}(0) &= \mathbf{1}, \end{aligned} \tag{E1}$$

is solved for Γ imaginary. Starting with the ansatz

$$U_{ij}(z) = u_{11}\exp(i\omega_{ij}z), \tag{E2}$$

one obtains a set of conditions

$$\begin{aligned} \omega_{21} &= \omega_{11} + \Omega, \\ \omega_{12} &= \omega_{22} - \Omega, \\ \frac{u_{21}}{u_{11}} &= \frac{i\omega_{11}}{\sigma\mu_0} = \frac{-\bar{\mu}_0}{i\omega_{21}}, \\ \frac{u_{12}}{u_{22}} &= \frac{i\omega_{22}}{-\bar{\mu}_0} = \frac{\sigma\mu_0}{i\omega_{12}}. \end{aligned} \tag{E3}$$

The second pair of these conditions defines the consistency conditions

$$\omega_{11}\omega_{21} = \omega_{12}\omega_{22} = \sigma|\mu_0|^2, \tag{E4}$$

which are converted into auxiliary equations:

$$\begin{aligned} \omega_{11}^2 + \Omega\omega_{11} - \sigma|\mu_0|^2 &= 0, \\ \omega_{22}^2 - \Omega\omega_{22} - \sigma|\mu_0|^2 &= 0. \end{aligned} \tag{E5}$$

Solutions to these quadratic equations are

$$\begin{aligned} \omega_{11\pm} = \omega_{12\pm} &= (-\Omega \pm \Xi)/2, \\ \omega_{21\pm} = \omega_{22\pm} &= (+\Omega \pm \Xi)/2, \end{aligned} \tag{E6}$$

where $\Xi :\equiv \tilde{\Gamma}\sqrt{q}/I$ and $q :\equiv q_5^2 + 4\sigma|Q|^2$. Then

$$U_{ij} = u_{ij+}\exp(iz\omega_{ij+}) + u_{ij-}\exp(iz\omega_{ij-}), \tag{E7}$$

and one has to solve the remaining conditions

$$\begin{aligned} \frac{u_{21\pm}}{u_{11\pm}} &= \frac{i\sigma\omega_{11\pm}}{\mu_0}, \\ \frac{u_{12\pm}}{u_{22\pm}} &= \frac{-i\omega_{22\pm}}{\bar{\mu}_0}, \end{aligned} \tag{E8}$$

in conjunction with the initial conditions

$$\begin{aligned} u_{11+} + u_{11-} &= 1, \\ u_{12+} + u_{12-} &= 0, \\ u_{21+} + u_{21-} &= 0, \\ u_{22+} + u_{22-} &= 1. \end{aligned} \tag{E9}$$

The solution is

$$\begin{aligned} u_{11+} &= -\frac{\omega_{11-}}{\Sigma}, \quad u_{11-} = +\frac{\omega_{11+}}{\Sigma}, \\ u_{22+} &= -\frac{\omega_{22-}}{\Sigma}, \quad u_{22-} = +\frac{\omega_{22+}}{\Sigma}, \\ u_{12+} &= -\frac{i\sigma\mu_0}{\Sigma}, \quad u_{12-} = +\frac{i\sigma\mu_0}{\Sigma}, \\ u_{21+} &= +\frac{i\bar{\mu}_0}{\Sigma}, \quad u_{21-} = -\frac{i\bar{\mu}_0}{\Sigma}, \end{aligned} \tag{E10}$$

leading to the final form:

$$\begin{aligned} U_{11} &= \exp\left(-i\frac{\Omega z}{2}\right)\left[\cos\left(\frac{\Xi z}{2}\right) + i\frac{\Omega}{\Xi}\sin\left(\frac{\Xi z}{2}\right)\right], \\ U_{12} &= i\frac{2\sigma Q}{\sqrt{q}}\exp\left(-i\frac{\Omega z}{2}\right)\sin\left(\frac{\Xi z}{2}\right), \\ U_{21} &= -i\frac{2\sigma\bar{Q}}{\sqrt{q}}\exp\left(i\frac{\Omega z}{2}\right)\sin\left(\frac{\Xi z}{2}\right), \\ U_{22} &= \exp\left(i\frac{\Omega z}{2}\right)\left[\cos\left(\frac{\Xi z}{2}\right) - i\frac{\Omega}{\Xi}\sin\left(\frac{\Xi z}{2}\right)\right]. \end{aligned} \tag{E11}$$

It is easy to check the unimodality condition $\det \mathbf{U}(z) = 1$ $(\forall z)$.

APPENDIX F: COMPONENTS OF Ω

The non-vanishing components of Ω are

$$
\begin{aligned}
\Omega^{B_1\bar{B}_3B_2} &= -\bar{\mu}\bar{\partial}_4\mu,\\
\Omega^{B_1\bar{B}_3B_3} &= -\bar{\mu}\bar{\partial}_1\mu,\\
\Omega^{B_1\bar{B}_3B_4} &= +\bar{\mu}\bar{\partial}_2\mu,\\
\Omega^{B_1\bar{B}_3\bar{B}_1} &= -\bar{\mu}\partial_3\mu,\\
\Omega^{B_1\bar{B}_3\bar{B}_2} &= -\mu\partial_4\mu,\\
\Omega^{B_1\bar{B}_3\bar{B}_3} &= +\mu\partial_1\mu,\\
\Omega^{B_1\bar{B}_3\bar{B}_4} &= +\bar{\mu}\partial_3\mu,\\
\Omega^{B_2\bar{B}_4B_1} &= -\mu\bar{\partial}_3\bar{\mu},\\
\Omega^{B_2\bar{B}_4B_3} &= +\bar{\mu}\bar{\partial}_1\bar{\mu},\\
\Omega^{B_2\bar{B}_4B_4} &= +\mu\bar{\partial}_2\bar{\mu},\\
\Omega^{B_2\bar{B}_4\bar{B}_1} &= -\bar{\mu}\partial_3\bar{\mu},\\
\Omega^{B_2\bar{B}_4\bar{B}_2} &= -\mu\partial_4\bar{\mu},\\
\Omega^{B_2\bar{B}_4\bar{B}_3} &= +\mu\partial_1\bar{\mu},\\
\Omega^{B_2\bar{B}_4\bar{B}_4} &= +\bar{\mu}\partial_2\bar{\mu}.
\end{aligned}
\tag{F1}
$$

APPENDIX G: LAGRANGE FORMULATION OF GENERALIZED HAMILTONIAN SYSTEMS

1. From the singular Lagrangian to the generalized Hamiltonian dynamics

The singular Lagrangian ($\mu \in \overline{1,D}$):

$$L = \mathcal{A}_\mu(x)\dot{x}^\mu - V(x), \tag{G1}$$

generates the Euler-Lagrange equations of motion:

$$\mathcal{E}_\mu \;:\equiv\; \frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} = F_{\mu\nu}\dot{x}^\nu - \partial_\mu V = 0, \tag{G2}$$

where $F_{\mu\nu} :\equiv \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ is the tensor of the "field strength". If it is nonsingular ($\det \mathbf{F} \neq 0$; possible only for even D), its inverse, the tensor of symplectic structure $\mathbf{J} :\equiv \mathbf{F}^{-1}$ can be defined. Then Eq. (G2) have the form of the generalized Hamilton equations:

$$\dot{x}^\mu = J^{\mu\nu}\partial_\nu V. \tag{G3}$$

To initiate the Hamilton formulation, one starts with the definition of canonical momenta

$$p_\mu :\equiv \frac{\partial L}{\partial \dot{x}^\mu} = \mathcal{A}_\mu(x), \tag{G4}$$

and of the (naively defined) Hamiltonian function

$$H :\equiv p_\mu \dot{x}^\mu - L. \tag{G5}$$

The definitions of momenta (G4) are supposed to be (non-singular) contact transformations replacing velocities $\dot{x}^\mu$ by the momenta p_ν, thus allowing (by inversion) to express the velocities $\dot{x}^\mu$ in terms of the momenta p_μ. Here, however, these equations are singular: the velocities do not figure (at all) on their right-hand-sides (RHS). Thus, instead of being the (successful) contact transformations, these equations are the constraints on the Hamiltonian dynamics of system:

$$\kappa_\mu :\equiv p_\mu - \mathcal{A}_\mu(x). \tag{G6}$$

Constraints obtained from the contact transformations are called **the primary constraints**, implying that there may be some additional constraints in the system (all these additional constraints are called **secondary constraints**). Hamiltonian systems with constraints are treated by the Dirac method which outlines are given in the rest of this Appendix.

For the specific system at hand, the naively defined Hamiltonian (G5) has the arbitrary weighted terms proportional to primary constraints:

$$H = V + \kappa_\mu \dot{x}^\mu \simeq V \tag{G7}$$

Its unique (non-arbitrary) part is called the **Canonical Hamiltonian** $H_c = V$. For the further purposes, one needs the (temptative) **Total Hamiltonian**:

$$H'_T = H_c + \kappa_\mu v^\mu. \tag{G8}$$

where v^μ are the **Lagrange multipliers**, for this moment taken to be arbitrary functions of time z. The prime on H'_T denotes the temptative nature of this quantity: once when (and if) the Lagrange multipliers are determined (i.e. replaced with suitable functions over phase space), this quantity will be replaced by symbol H_T. One can not know apriory what values will vs acquire. Instead, the (Dirac's) constraint analysis has to be performed to determine the full set of constraints in the system and then to classify the constraints as either **the first class** (ones that commute in a weak sense with all other constraints) or **the second class** (ones that are not of the first class). Then, the Lagrange multipliers standing next to the primary constraints of the second class will be determined as a specific functions on the phase space, while the Lagrange multipliers corresponding to the primary constraints of the first class will stay undetermined. Presence of the second class constraints leads to the reduction of the phase space, while the first class constraint generate the gauge transformations (on the phase space).

The phase space $\Gamma = \{(x,p) | \forall x, p \in \mathbb{R}^D\}$ possesses the **Poisson bracket**:

$$\{f,g\}_{PB} :\equiv \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu}. \tag{G9}$$

The evolution of any function f on the phase space Γ is defined by

$$\frac{df}{dt} = \{f, H'_T\}_{PB}. \tag{G10}$$

The consistency equations of the primary constraints

$$\frac{d\kappa_\mu}{dt} = \{\kappa_\mu, H'_T\}_{PB} = -\partial_\mu V + F_{\mu\nu}v^\nu = 0, \tag{G11}$$

can be solved if $\mathbf{F} = (F_{\mu\nu})$ is non-singular, giving the final expression for the Lagrange multipliers:

$$v^\mu = J^{\mu\nu}\partial_\nu V, \tag{G12}$$

If $\mathbf{F}$ is singular, some components of the RHS of Eq. (G11) can not be solved for v, and one has to define the secondary constrains (and then to check their consistency, and so on). In this work, only the case of non-singular $\mathbf{F}$ is discused.

Here, all primary constrains are of the second class, i.e. each of them has (at least) one other constraint that does not commute (in Poisson bracket sense) with it. This is easy to see from

$$\{\kappa_\mu, \kappa_\nu\}_{PB} = F_{\mu\nu}(x), \tag{G13}$$

and the non-singularity of the matrix $\mathbf{F}$. Constraints of the second class reduce the phase space of the system (whereas the first class constraints, if they existed, would be the gauge symmetries of the corresponding action).

To successfully perform the reduction of phase space, one needs to replace the Poisson brackets with the Dirac brackets, defined as:

$$\{f, g\}_{DB} \;:\equiv\; \{f, g\}_{PB} - \{f, \kappa_\mu\}_{PB} J^{\mu\nu} \{\kappa_\nu, g\}_{PB}\,. \tag{G14}$$

With respect to this structure, the connections κ are constant, i.e. every function $f(x,p)$ on the phase space commutes with them:

$$\{f, \kappa_\mu\}_{DB} = 0. \tag{G15}$$

On this way, one can consistently work with the reduced phase space $\Gamma^* :\equiv \Gamma/\{\kappa = 0\} = \{(x, p = \mathcal{A}(x)) | \forall x \in \mathbb{R}^D\}$. The Poisson bracket on this space is defined as

$$\{f, g\}_{PB\ on\ \Gamma^*} :\equiv \{f, g\}_{DB\ on\ \Gamma}|_{\kappa=0}\,. \tag{G16}$$

Since the coordinates x^μ do not commute with respect to the Dirac bracket

$$\{x^\mu, x^\nu\}_{DB} = J^{\mu\nu}, \tag{G17}$$

on the space Γ, their Poisson bracket on Γ^* are non-vanishing, too:

$$\{x^\mu, x^\nu\}_{PB\ on\ \Gamma^*} = J^{\mu\nu}|_{\Gamma^*}\,. \tag{G18}$$

From non-singularity of the tensor $\mathbf{J}$ one can conclude that the half of coordinates x can be used as the real coordinates on the Γ^*, and rest of them are the corresponding conjugated momenta.

2. When the generalized Hamiltonian dynamics has the Lagrangian formulation?

One can turn any even-dimensional generalized Hamiltonian system (given by Eqs (G3)) into the Lagrangian (G1) iff:

a) Matrix $\mathbf{J}$ is nonsingular, so one can define its inverse $\mathbf{F}$, and

b) 2-form $\hat{F} :\equiv \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ is closed, i.e. satisfies the Darboux condition $d\hat{F} = 0$.

The two conditions are (in a simple connected region U of the phase space: $\pi_1(U) = 0$) sufficient to assure the existence of "potentials" $\vec{\mathcal{A}} = \mathcal{A}_\mu dx^\mu$, such that $\hat{F}$ is exact 2-form $\hat{F} = d\vec{\mathcal{A}}$ (and, therefore, closed). Under these conditions the potentials are:

$$\mathcal{A}_\mu(x) = \frac{1}{2}\int_{x_0}^{x} F_{\mu\nu}(y)dy^\nu, \tag{G19}$$

where the integration is along any path connecting points x_0 and x, which belongs to the domain U, and x_0 is the point where $\mathcal{A}_\mu(x_0) = 0$.

The action has the form:

$$S[t_1, t_2] \;=\; \frac{1}{2}\int_{x_1}^{x_2} dx^\nu \int_{x_0}^{x} dy^\mu F_{\mu\nu}(y) - \int_{t_1}^{t_2} dt H(x(t)). \tag{G20}$$

The first term is the weighted surface integral over the surface spanned by points x_0, x_1 and x_2. The second term is the line term, i.e. it lives only on the line that connects the points x_1 and x_2.

[1] *M. Cronin-Golomb, J.O. White, B. Fisher, A. Yariv*; IEEE J. Quantum Electron. **QE-20**, 12 (1984).

[2] *M. Belić*, Phys. Rev. A**31**, 3169 (1985).

[3] *A. Bledowski, W. Krolikowski*, Opt. Lett. **13**, 146(1987).

[4] *A.A. Zozulya, V.T. Tikhonchuk*, Sov. J. Quantum Electron. **18**, 981 (1988).

[5] *D.A. Fish, A.K. Powell, T.J. Hall*, Opt. Commun. **88**, 28 (1992); **89**, 68 (1992).

[6] *P. Stojkov, M. Belić*, Phys. Rev. A**45**, 5061 (1992). Reprinted in **Landmark Papers on Photorefractive nonlinear Optics**, Eds: P. Yeh and C. Gu (World Scientific; 1995).

[7] *M. Belić, M. Petrović*, J. Opt. Soc. Am. B**11**, 481 (1994).

[8] *M. Berger*, **Geometrie** (Nathan, Paris, 1977).

[9] *K. Kowalski, W.H. Steeb*, Prog. Theor. Phys. **85**, 713 (1991).

[10] *V.I. Arnold ed.*, **Dynamical Systems I-IV** (Springer, Berlin, 1988-90).